9/27/2017 FIRST HOURLY **Math 21a, Fall 2017**

Name:

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MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 and 9, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T **F** The velocity vector of $\vec{r}(t) = \langle t, t, t \rangle$ at time $t = 2$ is twice the velocity vector at time $t = 1$.

Solution:
The velocity vector is the same at every time t.

2) T **F** The curvature of the curve $\vec{r}(t) = \langle 2t, 2t, 2t \rangle$ is always $1/2$.

Solution:
The curvature is zero because the curve is a line.

3) **T** F If $\vec{u} \times \vec{v} = \vec{0}$, then $\text{Proj}_{\vec{u}}(\vec{v}) \times \text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

Solution:
The projection vectors are parallel to the vectors onto which one has projected. Since the original vectors were parallel also the projections are parallel.

4) **T** F If $\vec{u} \times \vec{v} = \vec{v} \times \vec{w}$, then $\vec{v} \cdot (\vec{u} \times \vec{w}) = 0$.

Solution:
The assumption assumes that the three vectors u, v, w are in the same plane. The normal vector is also $u \times w$. But this is perpendicular to v.

5) T **F** There is a point not at the origin with Cartesian coordinates $(x, y, z) = (a, b, c)$ and spherical coordinates $(\rho, \theta, \phi) = (a, b, c)$.

Solution:
The assumption $a = \rho$ implies that only the first coordinate can be nonzero. If $z = 0$, then we have by assumption $\phi = 0$, but $\phi = 0$ and $z = 0$ implies $\rho = 0$ which is not possible.

6) T **F** The two planes $2x + 2y - z = 4$ and $-4x - 4y + 2z = 3$ intersect in a line.

Solution:
They are parallel and are not the same. So, they do not intersect at all.

7) | T | F | (T marked)

If the distance between two points P and Q is zero, then $P = Q$.

Solution:
If P and Q were different, then the points would be different.

8) | T | F | (F marked)

If the distance between two lines L and M is zero, then $L = M$.

Solution:
Lines can intersect without being equal. Their distance is then zero.

9) | T | F | (F marked)

The arc length of a circle with constant curvature κ is $2\pi\kappa$.

Solution:
It is $2\pi/r$.

10) | T | F | (F marked)

The surface $x^2 + y^2 + 4y = -z^2$ is a two-sheeted hyperboloid.

Solution:
Complete the square to get $x^2 - (y-2)^2 - z^2 = -4$

11) | T | F | (F marked)

There are two vectors $\vec{v}, \vec{w}$ in $\mathbf{R}^3$ of length 1 for which the dot product is 2.

Solution:
Cauchy-Schwarz forbids it

12) | T | F | (T marked)

If the acceleration of a curve $\vec{r}(t)$ is zero at all times and the velocity is non-zero at time $t = 0$, then the curve is a line.

13) T F The lines $\vec{r}(t) = \langle 3t, 4t, 5t \rangle$ and $\vec{s}(t) = \langle -4t, 3t, 0 \rangle$ intersect perpendicularly.

Solution:
The velocity vectors are perpendicular at the intersection point $(0, 0, 0)$.

14) T F The point given in spherical coordinates as $\rho = 2, \phi = \pi, \theta = \pi$ is on the z-axis.

Solution:
It is the south pole.

15) T F Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|\vec{u} \cdot \vec{v}||\vec{w}| = |\vec{u}||\vec{v} \cdot \vec{w}|$.

Solution:
Take $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \vec{w} = \langle 0, 1, 0 \rangle$.

16) T F The surface given in spherical coordinates as $\cos(\phi) = \rho$ is a cylinder.

Solution:
Multiply both by ρ to get $z = \rho^2$ which is a sphere.

17) T F The arc length of the curve $\langle \sin(t), \cos(t), t \rangle$ from $t = 0$ to $t = 2\pi$ is larger than 2π.

Solution:
It is a helix whose projection is a circle.

18) T F The surface parametrized by $\vec{r}(u, v) = \langle v \sin(u), v \cos(u), 0 \rangle$ with $0 \leq u < 2\pi$, $v \geq 0$ is a plane.

Solution:
Indeed it is the plane $z = 0$.

19) T F — It is possible that the intersection of an ellipsoid with a plane is a hyperbola.

Solution:
One is bounded, the other not.

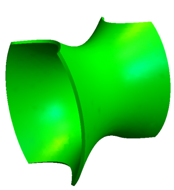
20) T F — For any two points P, Q and vectors $\vec{v}, \vec{w}$, the mid point $M = (P+Q)/2$ has the same distance to the two lines $\vec{r}_1(t) = P + t\vec{v}$ and $\vec{r}_2(t) = Q + t\vec{w}$.

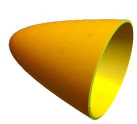
Solution:
This is not obvious and even the author of this problem (Oliver) got it first wrong. Yes, it is true that the mid point is on the plane equidistant to the two lines but the distance to the two lines can be different. Its best look at an extreme case. Let the first line be the z axes and the second line the line $\langle t, 2, 0\rangle$. Now if $P = (0,0,2)$ and $Q = (1000, 2, 0)$, then the midpoint is $(500, 1, 1)$

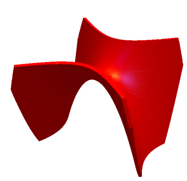
Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the surfaces with their equations $g(x,y,z) = 1$. Enter O, if there is no match.

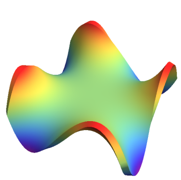

I


II


III

Function $g(x,y,z) =$	Enter O,I,II or III
$2x - y^2 - z^2$	
$2x^2 - y^2 + z^2$	
$2x - y$	
$2x^2 - y^2 - z$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.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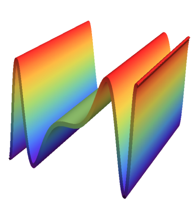
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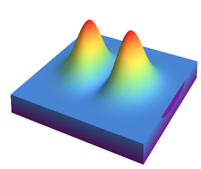
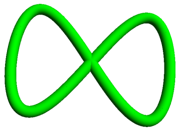

III

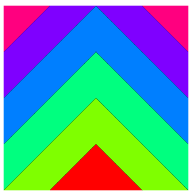
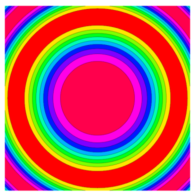
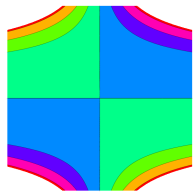
Function $f(x,y) =$	Enter O,I,II or III
$xy(x^2 - y^2)$	
$\sin(x^3)$	
$\sin(y^4)$	
$x^2 \exp(-x^2 - y^2)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

I  II  III 

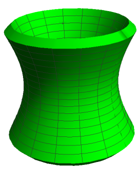
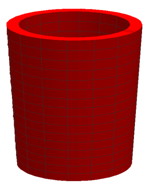
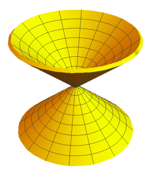
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t, \sin(4t), \cos(4t) \rangle$	
$\langle \cos(t), \cos(t), \sin(2t) \rangle$	
$\langle 3t, 1-t, 5t \rangle$	
$\langle t\sin(t), t\cos(t), t \rangle$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

I  II  III 

Function $g(x,y) =$	Enter O, I,II or III
$\sin(x^2+y^2)$	
$\sin(x) - y$	
$\|x\| + y$	
xy^2	

e) (2 points) Match the quadrics. Enter O if there is no match.

I  II  III 

Quadric	Enter O,I,II or III
$x^2+y^2-z^2=1$	
$x^2+y^2+z^2=1$	
$x^2+y^2=1$	
$x^2+y^2=z^2$	

Solution:
a) 2,1,0,3
b) 1,2,0,3
c) 3,2,0,1
d) 2,0,1,3
e) 1,0,2,3

Problem 3) (10 points)	(Only answers are needed)

a) (3 points) Write the equations of a surface in Cartesian, Cylindrical and Spherical coordinates. The first row gives an example:

Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
$z = 1$	$z = 1$	$\rho \cos(\phi) = 1$
		$\rho \sin(\phi) = 1$
	$r \cos(\theta) = 1$	
$x^2 + y^2 + z^2 = 1$		

b) (3 points) Assume $\vec{u}, \vec{v}$ are unit vectors which are perpendicular. Check one box in each row:

The value	is larger than 0	is smaller than 0	is equal to 0
$\vec{u} \cdot \vec{v}$			
$\lvert\vec{u} \times \vec{v}\rvert$			
$\vec{u} \cdot (\vec{v} \times \vec{u})$			

c) (2 points) Complete the following table which uses the vectors $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle, \vec{k} = \langle 0, 0, 1 \rangle, -\vec{i}, -\vec{j}, -\vec{k}$ or $\vec{0}$ in one of the first 3 boxes. Enter an scalar in each of the 3 boxes at the bottom.

$\vec{i} \times \vec{i} =$		$\vec{i} \times \vec{j} =$		$\vec{k} \times \vec{j} =$	
$\vec{j} \cdot \vec{i} =$		$\vec{j} \cdot \vec{j} =$		$\vec{j} \cdot \vec{k} =$	

d) (2 points) Complete the following table about the **TNB frame**. In each case, enter either $\vec{T}, \vec{N}, \vec{B}$ or $\vec{0}$ in each of the 6 boxes. Every correct row gives a point:

$\text{Proj}_{\vec{T}}(\vec{T}) =$		$\text{Proj}_{\vec{T}}(\vec{N}) =$		$\text{Proj}_{\vec{B}}(\vec{T}) =$	
$\text{Proj}_{\vec{N}}(\vec{T}) =$		$\text{Proj}_{\vec{N}}(\vec{N}) =$		$\text{Proj}_{\vec{N}}(\vec{B}) =$	

Solution:
a) The first row describes a cylinder. We have $\boxed{x^2+y^2=1}$, $\boxed{r=1}$.
The second row is the plane $\boxed{x=1}$ or $\boxed{\rho\sin(\phi)\cos(\theta)=1}$.
The third row is a sphere or $\boxed{r^2+z^2=1}$ or $\boxed{\rho=1}$.
b) Perpendicular vectors have a zero dot product $\boxed{0}$. The length of the cross product of two perpendicular vectors is the product of the lengths. It is $\boxed{\text{positive}}$. The triple scalar product is $\boxed{0}$ as two vectors are the same.
c) The first row is $\boxed{0}$,$\boxed{\vec{k}}$,$\boxed{-\vec{i}}$.
The second row is $\boxed{0}$,$\boxed{1}$,$\boxed{0}$.
d) It is useful here to note the following general facts: The projection of a vector onto a vector perpendicular is the zero vector.
The projection of a vector on itself is the vector itself.
We have used also that the vectors $\vec{T},\vec{N},\vec{B}$ are perpendicular.
In the first row we get $\boxed{\vec{T}}$,$\boxed{\vec{0}}$,$\boxed{\vec{0}}$.
In the second row we get $\boxed{\vec{0}}$,$\boxed{\vec{N}}$,$\boxed{\vec{0}}$.

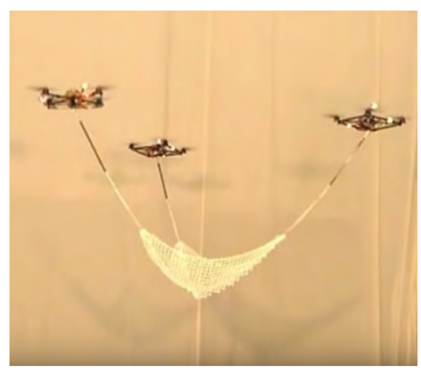
Problem 4) (10 points)

In a TED talk of 2013, **Raffaello D'Andrea** and a team demonstrated **"quadrotor athletes"**.

Assume the three robots are at positions

$$A=(2,3,1), B=(2,3,3) \text{ and } C=(5,4,2).$$

What is the area of the triangle they span?



A figure from an article by J. Sidman and A. St John in the Notices of the AMS, October 2017.

Solution:
We compute the area of the parallel epiped as $\vec{BC}\times\vec{AC}=\langle -2,6,0\rangle$ which has length $\sqrt{40}$. The area of the triangle is $\boxed{\sqrt{10}}$.

Problem 5) (10 points)

The six-legged **Gough-Stewart platform** has applications in flight simulators, robotics, crane technology, underwater research, telescopes and orthopedic surgery. The bottom positions of the legs are

$$A_1 = (5, -2, 0), A_2 = (5, 1, 0), A_3 = (-3, 5, 0),$$

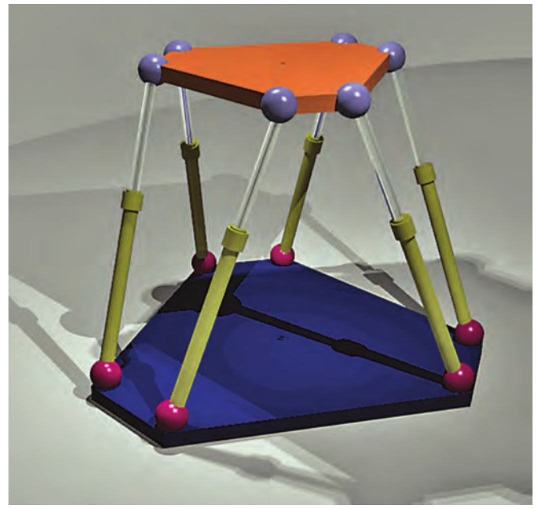
$$A_4 = (-5, 3, 0), A_5 = (-5, -3, 0), A_6 = (-3, -5, 0).$$

The top positions of the legs are

$$B_1 = (-5, 2, 6), B_2 = (-5, -1, 6), B_3 = (3, -5, 6),$$

$$B_4 = (5, -3, 6), B_5 = (5, 3, 6), B_6 = (3, 5, 6).$$

a) (5 points) What is the distance between B_1 and the plane containing $A_1, \dots, A_6$?
b) (5 points) What is the distance between B_1 and the line through A_1 and A_2?



Picture: Jessica Sidman and Audry St. John in the Notices of the AMS

Solution:
a) The distance 6 can be read off directly as all the points A_k are in the $z = 0$ plane and all the B_k are at $z = 6$.
It is also possible to use the **distance formula**: find the normal vector $\langle 0, 0, 1\rangle$, then form $|\vec{B_1A_1} \cdot \vec{n}|/|n| = 6/1 = 6$ which has an interpretation "Volume"/"area".
b) We can use the **distance formula between a point and a line**: It is the area of the parallelogram spanned by $\vec{v} = \vec{A_1A_2} = \langle 0, 3, 0\rangle$ and $\vec{w} = \vec{A_1B_1} = \langle -10, -4, 6\rangle$, divided by the length of $\vec{v}$. This is is $|\langle 18, 0, 30\rangle|/|\langle 0, 3, 0\rangle| = \sqrt{18^2 + 30^2}/3 = \boxed{2\sqrt{34}}$.

Problem 6) (10 points)

Even though Saturn is much larger than the Earth, its gravitational force is only 7 percent larger than here on Earth. When **Cassini** plunged into Saturn, it felt an acceleration

$$\vec{r}\,''(t) = \langle \pi \sin(\pi t), 0, -10 - 2t\rangle \ .$$

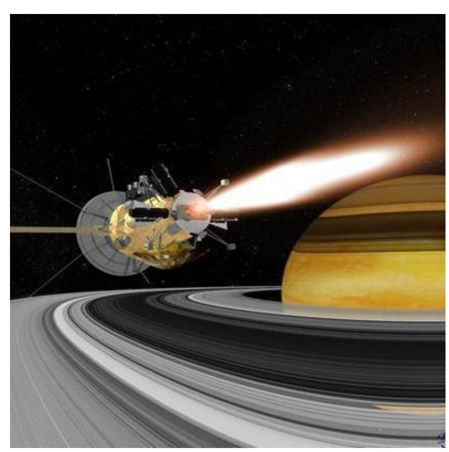
We know the initial velocity

$$\vec{r}\,'(0) = \langle 2, 5, 1\rangle$$

and initial position

$$\vec{r}(0) = \langle 0, 0, 1000\rangle \ .$$

Where is the spacecraft at $t = 1$?



Picture: NASA

Solution:
Integrate:

$$\vec{r}\,'(t) = \langle -\cos(\pi t), 0, -10t - t^2 \rangle + \langle C_1, C_2, C_3 \rangle \ .$$

Fixing the constants at $t = 0$ gives

$$\vec{r}\,'(t) = \langle -\cos(\pi t) + 3, 5, 1 - 10t - t^2 \rangle \ .$$

Integrate again to get

$$\vec{r}\,(t) = \langle -\sin(\pi t)/\pi + 3t + C_1, 5t + C_2, t - 5t^2 - t^3/3 + C_3 \rangle \ .$$

Fix the constants to get

$$\vec{r}\,(t) = \langle -\sin(\pi t)/\pi + 3t, 5t, 1000 + t - 5t^2 - t^3/3 \rangle \ .$$

At $t = 1$, we are at the point $(3, 5, 996 - 1/3)$ which is $\boxed{(3, 5, 2987/3)}$.

Problem 7) (10 points)

Let's look at the two planes

$$x + y + z = 1$$

and

$$x + y - z = 1 \ .$$

a) (4 points) Find the plane $ax+by+cz = d$ through $P = (1, 0, 0)$ which is perpendicular to both.

b) (2 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1, 0, 0)$ which is contained in both planes.

c) (4 points) Find a parametrization $\vec{r}(t)$ of a line through $P = (1, 0, 0)$ which is contained in the first plane but not the second.

Solution:
a) To get the normal vector, take the cross product between the normal vector $\langle 1, 1, 1 \rangle$ and $\langle 1, 1, -1 \rangle$ of the two planes. This is $\langle -2, 2, 0 \rangle$. The equation of the plane is $-2x + 2y = d$, where d is a constant. Plugging in a point gives $\boxed{-2x + 2y = 2}$.
b) The parametrization is $\vec{r}\,(t) = \langle 1, 0, 0 \rangle + t\langle -2, 2, 0 \rangle$ which is $\boxed{\vec{r}(t) = \langle t + 1, -t, 0 \rangle}$.
c) Pick a point Q on the first plane but not on the second. A possibility is $Q = (0, 0, 1)$. The vector $\vec{PQ}$ is $\langle -1, 0, 1 \rangle$. The parametrization is $\vec{r}\,(t) = \langle 1, 0, 0 \rangle + t\langle -1, 0, 1 \rangle$ which is $\boxed{\vec{r}(t) = \langle 1 - t, 0, t \rangle}$. Of course, there would be other parametrizations, depending on the point chosen. A commonly seen solution is also $\langle 1 + t, 0, -t \rangle$.

Problem 8) (10 points)

The world was supposed to end on September 23 due to the mysterious planetary system HD 7924. But here you sit and have to take the first hourly. A moon on HD 7924 moves on an epicycle

$$\vec{r}(t) = \langle 10\cos(t), 10\sin(t), 0\rangle + \langle 2\cos(5t), 2\sin(5t), 0\rangle \ .$$

a) (2 points) Find the velocity $\vec{r}\,'(0)$ at $t = 0$ and the speed $|\vec{r}\,'(0)|$ at $t = 0$.

b) (2 points) Find the acceleration $\vec{r}\,''(0)$ at $t = 0$.

c) (3 points) Find $\kappa(0) = |\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$.

d) (3 points) Inhabitants from HD 7924 beam you the hint $|\vec{r}\ '(t)|^2 = 400\cos^2(2t)$. Use this to find the arc length from $t = 0$ to $t = 2\pi$.

Will the world end on September 23?

30 Updated on September 20, 2017 at 12:37 PM, Posted on September 20, 2017 at 11:03 AM

Viral videos and theories claim an apocalypse will begin on Sept. 23. This photo is an artist's impression of a view from the HD 7924 planetary system looking back toward our sun. (Karen Teramura; BJ Fulton, University of Hawaii, Institute for Astronomy.)

Fun fact: the guy who came up with the date September 23 has revised his estimate to October 15. There is still hope to avoid the second midterm ...

Solution:

a) The velocity is

$$\vec{r}\,'(t) = \langle -10\sin(t), 10\cos(t), 0\rangle + \langle -10\sin(5t), 10\cos(5t), 0\rangle$$

Evaluated at $t = 0$, this is $\langle 0, 20, 0\rangle$. The length of this vector is $\boxed{20}$.

b) Differenticating again gives

$$\vec{r}\,''(t) = \langle -10\cos(t), -10\sin(t), 0\rangle + \langle -100\cos(5t), -100\sin(5t), 0\rangle$$

At $t = 0$, this is $\boxed{\langle -60, 0, 0\rangle}$.

c) The cross product is $\langle 0, 0, 1200\rangle$. It length is 1200. Divided by 20^3 gives $\boxed{3/20}$.

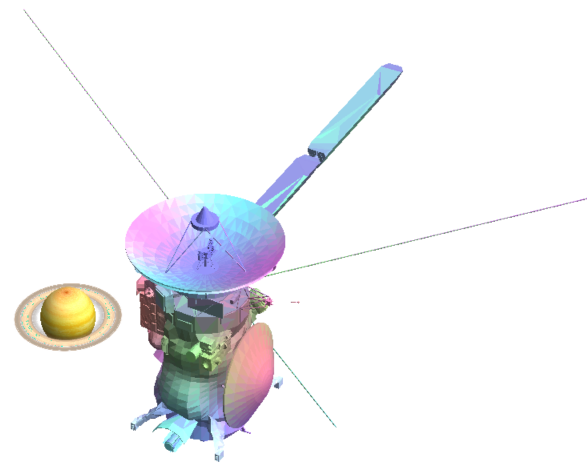
d) Since the speed is $20|\cos(t)|$, we get

$$20\int_0^{2\pi} |\cos(t)|\ dt = 20 \cdot 2\int_{-\pi/2}^{\pi/2} \cos(t)\ dt = 20 \cdot 2 \cdot 2$$

which is $\boxed{80}$. A common mistake was to integrate $\cos(t)$ blindly which gives zero. An arc length is always positive if a curve is not just degenerated to a point.

Problem 9) (10 points)

Two weeks ago, in a grand finale, the Cassini space craft plunged into the atmosphere of **Saturn**. To build a model of the situation we have to parametrize various parts on the probe which were used both for measurement and communication.
You don't have to specify the parameter bounds but give the parametrizations for each of the 5 objects:



Picture: by Mathematica using a printable

3D STL models provided by NASA

a) (2 points) Saturn is a sphere $(x-1)^2 + y^2 + z^2 = 16$.

$$\vec{r}(\theta,\phi) = \langle \quad , \quad , \quad \rangle .$$

b) (2 points) The rings are given by $z = 0, r^2 = x^2 + y^2 \leq 25$.

$$\vec{r}(r,\theta) = \langle \quad , \quad , \quad \rangle .$$

c) (2 points) The satellite dish $(x-50)^2 + (y-70)^2 = z$ beams pictures back to earth.

$$\vec{r}(x,y) = \langle \quad , \quad , \quad \rangle .$$

d) (2 points) There is also a satellite antenna of the form $(x-50)^2 + z^2 = 1/100$.

$$\vec{r}(\theta,y) = \langle \quad , \quad , \quad \rangle .$$

e) (2 points) There is also a device of the form $(x-50)^2 + z^2 - (y-70)^2 = 1$.

$$\vec{r}(\theta,y) = \langle \quad , \quad , \quad \rangle .$$

Solution:

a) Translate the **sphere** of radius 4 by $\langle 1,0,0\rangle$:

$\boxed{\langle 1+4\cos(\theta)\sin(\phi), 4\sin(\theta)\sin(\phi), 4\cos(\phi)\rangle}$.

b) This is the xy-plane parametrized using polar coordinates:

$\boxed{\langle r\cos(\theta), r\sin(\theta), 0\rangle.}$.

c) This is a **paraboloid**. We can either translate the paraboloid by $\langle 50,70,0\rangle$:

$\boxed{\langle 50+\sqrt{z}\cos(\theta), 70+\sqrt{z}\sin(\theta), z\rangle}$.

An other possibility is to use the standard x and y coordinates and get $\boxed{\langle x, y, (x-50)^2+(y-70)^2\rangle}$.

d) This is a **hyperbolic paraboloid** translated by $\langle 50,70,0\rangle$. We can write $\langle 50,70,0\rangle + \langle\sqrt{1+y^2}\cos(\theta), y, \sqrt{1+y^2}\sin(\theta)\rangle = \boxed{\langle 50+\sqrt{1+y^2}\cos(\theta), 70+y, \sqrt{1+y^2}\sin(\theta)\rangle}$.

In that solution y is a translated y-coordinate. An other possibility is

$\langle 50+\sqrt{1+(y-70)^2}\cos(\theta), y, \sqrt{1+(y-70)^2}\sin(\theta)\rangle$