Homework 9: Functions and Continuity

This homework is due Friday, 10/2 rsp Tuesday 10/6.

1 Which of the following functions are continuous everywhere on the real line? No reasoning is required. Each question one point. The $\log = \ln$ is the natural log.

a) $(x^2 - 4)/(x - 2)$.
b) $\log(|x|)$
c) $1/\log(|\sin(x)/2|)$
d) $1/\cos(x)$
e) $1/\log(2 + |x|)$

f) $\log(\exp(x))$
g) $\exp(1/x)$
h) $\sin(e^{\cos(x)})$
i) $\exp(1/x)$
j) $\exp(\log|x|)$

2 Investigate whether the following functions are continuous at $(0,0)$. If the limit $(x, y) \to (0,0)$ exists, tell what the limit is.
a) $f(x, y) = x^2/(x^3 + y^3)$. b) $f(x, y) = x^2y^2/(x^2 + y^2)$.

It can be helpful to use polar coordinates. You have to give some reasoning here.

3 Find the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ if it exists or show that the limit does not exist
a) $f(x, y) = \frac{6x^4y}{2x^5 + y^5}$ b) $f(x, y) = \frac{x^6 - y^6}{(x^2 + y^2)^2}$
Also here is some reasoning required.

4 Determine the set of points where the following function is continuous
$$f(x, y) = \frac{e^x + e^y}{e^{xy} - 1}.$$

5 Find the limit $(x, y) \to (0, 0)$ of the function $f(x, y) = \sin(x^2 + y^2) \log(x^2 + y^2)$, where $\log(x) = \ln(x)$.

Main definitions

A function $f(x, y)$ with domain R is **continuous at** $(a, b) \in R$ if $f(x, y) \to f(a, b)$ for all choices $(x, y) \to (a, b)$. We even say is **continuous at a point** (a, b) **not in the domain** if there exists a finite value $f(a, b)$ such that $f(x, y) \to f(a, b)$ whenever $(x, y) \to (a, b)$ for $(x, y) \in R$. The function $f(x, y) = y(x^2 - 1)/(x - 1)$ is continuous everywhere even so $x = 1$ is not in the domain. Also $f(x, y) = \sin(x^2 + y^2)/(x^2 + y^2)$ is continuous as l'Hopital for polar coordinates shows by setting $f(0, 0) = 1$, it becomes continuous even at the point $(0, 0)$. In one dimensions, there are **jump discontinuities** like $f(x) = \text{sign}(x)$ or **poles** like $f(x) = 1/x$ or **oscillations** like $f(x) = \sin(1/x)$. These three prototypes can occur together in examples like $1/\sin(1/x))$ or $\arctan(1/\sin(1/x))$. Many questions about continuity in two dimensions are answered when writing the function in polar coordinates $x = r \cos(\theta), y = r \sin(\theta)$ near the point in question. For $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ for example, the function becomes (just fill in $x = r \cos(\theta), y = r \sin(\theta)$), in polar coordinates $f(r, \theta) = \cos(2\theta)$. The value of the function depends only on the angle. Arbitrarily close to $(0, 0)$, the function takes any values between -1 and 1. The function is not continuous because we can not find a value at $(0, 0)$ such that it can be continuously extended to it.