

## Homework 9: Functions and Continuity

This homework is due Friday, 10/2 resp Tuesday 10/6.

- 1 Which of the following functions are continuous everywhere on the real line? No reasoning is required. Each question one point. The  $\log = \ln$  is the natural log.

- |                          |                        |
|--------------------------|------------------------|
| a) $(x^2 - 4)/(x - 2)$ . | f) $\log(\exp(x))$     |
| b) $\log( x )$           | g) $\exp(1/x)$         |
| c) $1/\log( \sin(x)/2 )$ | h) $\sin(e^{\cos(x)})$ |
| d) $1/\cos(x)$           | i) $\exp(1/x)$         |
| e) $1/\log(2 +  x )$     | j) $\exp(\log x )$     |

### Solution:

- a) Continuous
- b) Not continuous
- c) Continuous
- d) Not Continuous.
- e) Continuous everywhere
- f) Continuous everywhere
- g) Not continuous at  $x = 0$
- h) Continuous
- i) Not continuous j) Continuous

- 2 Investigate whether the following functions are continuous at  $(0, 0)$ . If the limit  $(x, y) \rightarrow (0, 0)$  exists, tell what the limit is.
- a)  $f(x, y) = x^2/(x^3 + y^3)$ . b)  $f(x, y) = x^2y^2/(x^2 + y^2)$ .

It can be helpful to use polar coordinates. You have to give some reasoning here.

**Solution:**

a) Not continuous. Put  $y = 0$  and take the limit  $x \rightarrow 0$ .

b) Continuous. Use polar coordinates to see that it is  $r^2 \sin^2(\theta) \cos^2(\theta)$ .

3 Find the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  if it exists or show that the limit does not exist

a)  $f(x,y) = \frac{6x^4y}{2x^5+y^5}$       b)  $f(x,y) = \frac{x^6-y^6}{(x^2+y^2)^2}$

Also here is some reasoning required.

**Solution:**

The limit equals to 0. Claim: as  $(x,y) \rightarrow (0,0)$ ,

$$\lim \frac{x^6}{(x^2 + y^2)^2} \rightarrow 0.$$

Indeed,

$$\left| \frac{x^6}{(x^2 + y^2)^2} \right| \leq \left| \frac{x^6}{(x^2)^2} \right| \leq x^2.$$

Since,  $x^2 \rightarrow 0$  as  $x \rightarrow 0$ , this proves the claim. By the same reason,

$$\lim \frac{y^6}{(x^2 + y^2)^2} \rightarrow 0.$$

It follows that the difference

$$\lim \frac{x^6 - y^6}{(x^2 + y^2)^2} \rightarrow 0.$$

- 4 Determine the set of points where the following function is continuous

$$f(x, y) = \frac{e^x + e^y}{e^{xy} - 1}.$$

**Solution:**

The numerator and the denominator are continuous functions of  $x$  and  $y$ ; hence the only problems in continuity can arise when the denominator equals to 0. This occurs when  $e^{xy} = 1$  or  $xy = 0$  forcing either  $x = 0$  or  $y = 0$ . The numerator is always positive, so we never have to worry about whether the limit of  $0/0$  exists. Hence,  $H(x, y)$  is continuous at all points in the plane except for the  $x$ -axis and the  $y$ -axis.

- 5 Find the limit  $(x, y) \rightarrow (0, 0)$  of the function  $f(x, y) = \sin(x^2 + y^2) \log(x^2 + y^2)$ , where  $\log(x) = \ln(x)$ .

**Solution:**

First notice that as  $(x, y) \rightarrow 0$ ,  $x^2 + y^2 \rightarrow 0$ . Setting  $x^2 + y^2 = r^2$ , it suffices to find the limit as  $r \rightarrow 0$  of  $\sin r \cdot \log r$ . As  $r \rightarrow 0$ ,  $\sin r \rightarrow 0$  and  $\log r \rightarrow -\infty$ ; so we have to be a bit careful because we are dealing with  $0 \cdot (-\infty)$ . One can use l'Hôpital's rule: write  $u = r^2$  and take the limit

$$\frac{\log(u)}{(1/\sin(u))}$$

for  $u \rightarrow 0$ . By differentiating top and bottom we get the limit

$$(1/u)/(-\cos(u)/\sin^2(u)) = -(\sin(u)/u)(\sin(u)/\cos(u))$$

The first term goes to 1 by l'Hopital. The second goes to zero. The limit is zero.

**Main definitions**

A function  $f(x, y)$  with domain  $R$  is **continuous at**  $(a, b) \in R$  if  $f(x, y) \rightarrow f(a, b)$  for all choices  $(x, y) \rightarrow (a, b)$ . We even say is **continuous at a point**  $(a, b)$  **not in the domain** if there exists a finite value  $f(a, b)$  such that  $f(x, y) \rightarrow f(a, b)$  whenever  $(x, y) \rightarrow (a, b)$  for  $(x, y) \in R$ . The function  $f(x, y) = y(x^2 - 1)/(x - 1)$  is continuous everywhere even so  $x = 1$  is not in the domain. Also  $f(x, y) = \sin(x^2 + y^2)/(x^2 + y^2)$  is continuous as l'Hopital for polar coordinates shows by setting  $f(0, 0) = 1$ , it becomes continuous even at the point  $(0, 0)$ . In one dimensions, there are **jump discontinuities** like  $f(x) = \text{sign}(x)$  or **poles** like  $f(x) = 1/x$  or **oscillations** like  $f(x) = \sin(1/x)$ . These three prototypes can occur together in examples like  $1/\sin(1/x)$  or  $\arctan(1/\sin(1/x))$ . Many questions about continuity in two dimensions are answered when writing the function in polar coordinates  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$  near the point in question. For  $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$  for example, the function becomes (just fill in  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ ), in polar coordinates  $f(r, \theta) = \cos(2\theta)$ . The value of the function depends only on the angle. Arbitrarily close to  $(0, 0)$ , the function takes any values between  $-1$  and  $1$ . The function is not continuous because we can not find a value at  $(0, 0)$  such that it can be continuously extended to it.