Homework 6: Arc length and curvature

This homework is due Wednesday, 9/23 rsp Thursday 9/24.

1 a) Find the arc length of the curve $\vec{r}(t) = \langle t^3, 24t, 6t^2 \rangle$, where $t \in [0, 10]$.
b) The shadow of the curve on the xz-plane is a curve in two dimensions given by $\vec{r}(t) = \langle t^3, 6t^2 \rangle$ with $t \in [0, 10]$. Find this length. You are in trouble if this number is larger than the result in a).

2 Arc length can be defined in any dimensions. A curve in 4 dimension is parametrized as $\vec{r}(t) = \langle x_1(t), x_2(t), x_3(t), x_4(t) \rangle$. Find the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), 1/t, \log(t) \rangle \ ,$$

where $\log(t) = \ln(t)$ is the natural log and $1 \leq t \leq 4$.

3 a) Use a calculator, Mathematica or Wolfram alpha to evaluate the arc length of the curve $\vec{r}(t) = \langle t^3, t^6, t^9 \rangle$ from $t = 0$ to $t = 2$.
b) Do the same with $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $t = 0$ to $t = 8$. Compare with the result in a) and explain in 3 words why you got the same result.

4 Use the formula

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

to compute the curvature $\kappa(t)$ of

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

at $t = 1$.

5 Find the parameter c such that the parabola $y = cx^2$ has curvature 30 at the origin.

Main definitions

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then

$$L = \int_a^b |\vec{r}\,'(t)| \, dt$$

is called the **arc length of the curve**. Written out in coordinates, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we have

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt \ .$$

For curves in two dimensions, where $\vec{r}(t) = \langle x(t), y(t) \rangle$ has two coordinates only, we have $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, dt$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} \ .$$

The curvature of a circle of radius r is equal to $1/r$ at every point of the circle. The curvature is zero for a line.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \ .$$