

Homework 6: Arc length and curvature

This homework is due Wednesday, 9/23 resp Thursday 9/24.

- 1 a) Find the arc length of the curve $\vec{r}(t) = \langle t^3, 24t, 6t^2 \rangle$, where $t \in [0, 10]$.
- b) The shadow of the curve on the xz -plane is a curve in two dimensions given by $\vec{r}(t) = \langle t^3, 6t^2 \rangle$ with $t \in [0, 10]$. Find this length. You are in trouble if this number is larger than the result in a).

Solution:

- a) We have $|\vec{r}'(t)| = 3(8 + t^2)$. Integrating this over the interval $[0, 10]$ gives 1240.
- b) We have $\int_0^{10} 3t\sqrt{t^2 + 16} dt$. Use substitution. We get 198.528. $8(29^{3/2} - 8) = 1185.36$.

- 2 Arc length can be defined in any dimensions. A curve in 4 dimension is parametrized as $\vec{r}(t) = \langle x_1(t), x_2(t), x_3(t), x_4(t) \rangle$. Find the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), 1/t, \log(t) \rangle,$$

where $\log(t) = \ln(t)$ is the natural log and $1 \leq t \leq 4$.

Solution:

We have to integrate $(1+t^2)/t^2$ from $t = 1$ to $t = 4$. The answer is $15/4$.

- 3 a) Use a calculator, Mathematica or Wolfram alpha to evaluate the arc length of the curve $\vec{r}(t) = \langle t^3, t^6, t^9 \rangle$ from $t = 0$ to $t = 2$.
b) Do the same with $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $t = 0$ to $t = 8$. Compare with the result in a) and explain in 3 words why you got the same result.

Solution:

- a) The arc length equals to $\int_0^2 \sqrt{9t^4 + 36t^{10} + 81t^{16}} dt$. Numerical integration gives us ~ 517.604 .
b) the result is the same because we have a different parametrization.

- 4 Use the formula

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

to compute the curvature $\kappa(t)$ of

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

at $t = 1$.

Solution:

Differentiating, we find: $\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ and $\vec{r}''(t) = \langle 0, 2, 6t \rangle$. At $t = 1$, $\vec{r}'(1) = \langle 1, 2, 3 \rangle$, while $\vec{r}''(1) = \langle 0, 2, 6 \rangle$. The cross-product is $\vec{r}'(1) \times \vec{r}''(1) = \langle \dots \rangle$. This is $\sqrt{36 + 36 + 4}/\sqrt{14}^3$.

- 5 Find the parameter c such that the parabola $y = cx^2$ has curvature 30 at the origin.

Solution:

The curvature of a plane curve is given by

$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

Letting $t = x$ be the parameter, we find that

$$\kappa = \frac{2c}{(1 + 4c^2x^2)^{3/2}}.$$

At $t = x = 0$, we see that $\kappa = 2c$ so for κ to be 30, we need to take the parameter $c = 15$. We also can have $c = -15$ but this is the same parabola just flipped around.

Main definitions

If $t \in [a, b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}'(t)$ and speed $|\vec{r}'(t)|$, then

$$L = \int_a^b |\vec{r}'(t)| dt$$

is called the **arc length of the curve**. Written out in coordinates, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we have

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt .$$

For curves in two dimensions, where $\vec{r}(t) = \langle x(t), y(t) \rangle$ has two coordinates only, we have $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t , then we can define $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**.

The **curvature** of a curve at the point $\vec{r}(t)$ is defined as

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}.$$

The curvature of a circle of radius r is equal to $1/r$ at every point of the circle. The curvature is zero for a line.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$