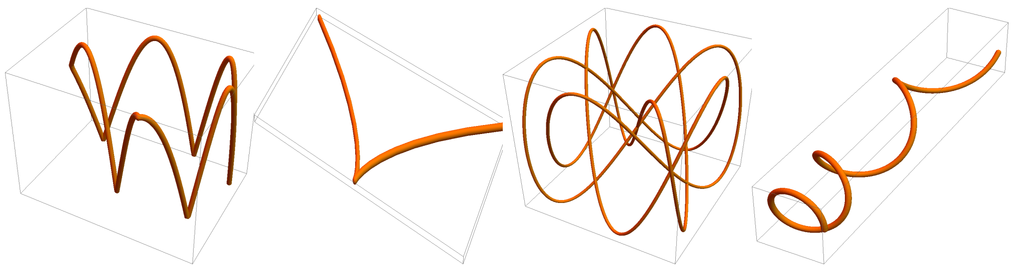
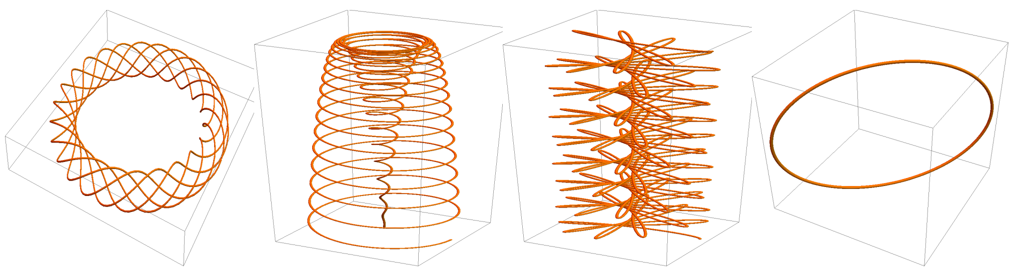
Homework 5: Parametrized curves

This homework is due Monday, 9/21 rsp Tuesday 9/22.

1 Match the curves:



I II III IV



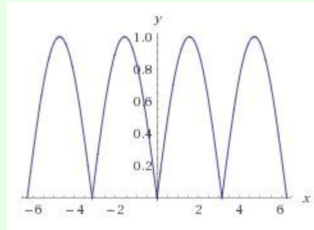
V VI VII VIII

$\vec{r}(t) =$	I-VIII
$\langle t\cos(8t), t\sin(8t), t(8\pi - t)\rangle$	
$\langle (6+\cos(24t))\cos(5t), (6+\cos(24t])\sin(5t), \sin(24t)\rangle$	
$\langle \cos(t40)\cos(3t), \cos(t40)\sin(4t), t\rangle$	
$\langle \cos(t), t^2, \sin(t)\rangle$	
$\langle t^3, t^2, 0\rangle$	
$\langle \cos(3t), \sin(4t), \cos(7t)\rangle$	
$\langle \cos(t), \cos(t), \sin(t)\rangle$	
$\langle \lvert\cos(t)+\sin(t)\rvert, \lvert\sin(t)\rvert, \lvert\cos(5t)\rvert\rangle$	

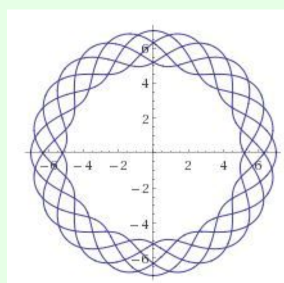
Solution:

VI,V,IV,VII,IV,II,III,VIII,I

Solution:
IV, continued)



The graph of $|\cos(5t)|$ looks similar, but with a smaller period. These 'bounces' are reflected in graph IV.
V) This corresponds to the parametrization
$\vec{r}(t) = \langle (6 + \cos(24t)) \cos(5t), (6 + \cos(24t)) \sin(5t), \sin(24t) \rangle$. We can see that the image in the xy-plane will be circular. The radius of this circle will be varying periodically with t, ranging from 5 when $\cos(24t) = -1$ to 7 when $\cos(24t) = 1$. Thus, the image will look like this:



And this corresponds to graph V.
VI) This corresponds to the parametrization
$\vec{r}(t) = \langle t\cos(8t), t\sin(8t), t(8\pi - t) \rangle$. We can see that the image in the xy-plane is circles, whose radius gets larger as t increases. Meanwhile, the $z(t) = 8\pi t - t^2$. This describes a parabola that increases until reaching a maximum and then decreases. Putting these behaviors together leads us to choose graph VI.

Solution:
VII) This corresponds to the parametrization
$\vec{r}(t) = \langle \cos(40t)\cos(3t), \cos(40t)\sin(4t), t\rangle$. We see that $z(t) = t$, so we can expect that the curve will increase in z linearly as t increases. This points us to graph VII. As further confirmation, we can see that the $\cos(3t)$ component in $x(t)$ and the $\sin(4t)$ component in $y(t)$ will lead to an image in the xy-plane that will look something like a flower (like in part III), which is reflected in graph VII.
VIII) This corresponds to the parametrization
$\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t)\rangle$. The image in the xz-plane is a circle and so is the image in the yz-plane. This describes graph VIII.

2 Parametrize the intersection of the elliptic paraboloid $z = 2x^2+y^2$ with the quintic cylinder $y = 1 + x^5$.

Solution:
Start with the second equation and set $x = t$. Then, use the equations given to find y and z: $y = 1+t^5$ and $z = 2t^2+(1+t^5)^2$. We end up with

$$\vec{r}(t) = \langle t, 1 + t^3, 2t^2 + (1 + t^5)^2\rangle \ ,$$

for $t \in R$.

3 a) Two particles travel along the space curves $\vec{r}_1(t) = \langle t, t^2, t^3\rangle$ and $\vec{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t\rangle$. Do the particles collide? Do the particle paths intersect?
b) If $\vec{r}(t) = \langle \cos(t), 2\sin(t), 4t\rangle$, find $\vec{r}\ ''(0)$ and $\vec{r}\ '(0)$. Then compute $|\vec{r}\ '(0) \times \vec{r}\ ''(0)|/|\vec{r}\ '(0)|^3$. We will later call this the curvature.

Solution:

(a) If they collide, the two particles must be at the same position at the same time. This gives us three equations: $t = 1 + 2t$, $t^2 = 1+6t$, and $t^3 = 1+14t$. Solving the first one gives $t = -1$. However, plugging this value of t into the second equation is not a solution, as $1 \neq 7$. Therefore, there is no collision.

If they intersect, the two particles must be at the same position at different times. This gives us the three equations: $t = 1+2s$, $t^2 = 1 + 6s$ and $t^3 = 1 + 14s$. We can solve them to get $s = 0, t = 1$ or $s = \frac{1}{2}, t = 2$. Taking the first solution, when $s = 0$ and $t = 1$, this gives us a path intersection at the point $(1, 1, 1)$. Taking the second solution, when $s = 0$ and $t = 1$, this gives us a path intersection at the point $2, 4, 8$.

(b) We compute:

$$\vec{r}\,'(0) = \langle -\sin(0), 2\cos(0), 4\rangle = \langle 0, 2, 4\rangle$$

$$\vec{r}\,''(0) = \langle -\cos(0), -2sin(0), 0\rangle = \langle -1, 0, 0\rangle.$$

Next, we compute the curvature at 0:

$$\frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{|\langle 0, -4, 2\rangle|}{\sqrt{20}^3} = \frac{\sqrt{20}}{\sqrt{20}^3} = \frac{1}{20}.$$

4 Find the point of intersection of the two tangent lines to the curve $\vec{r}(t) = \langle \sin(\pi t), 2\sin(\pi t), \cos(\pi t)\rangle$ at the points where $t = 0$ and $t = 0.5$.

Solution:
We parametrize the two lines as

$$\langle 0,0,1\rangle + t\langle \pi, 2\pi, 0\rangle$$

and

$$\langle 1,2,0\rangle + s\langle 0,0,-\pi\rangle \ .$$

To find the point of intersection, we solve for t and s such that $\pi t = 1$, $2\pi t = 2$, and $1 = -\pi s$. We solve this to get that the paths intersect when $s = -1/\pi$ and $t = 1/\pi$ at the point (1,2,1).

5 A particle moving along a curve $\vec{r}(t)$ has the property that $\vec{r}\,''(t) = \langle 1, 0, \sin(2t)\rangle$. We know $\vec{r}(0) = \langle 1,1,2\rangle$ and $\vec{r}\,'(0) = \langle 1,0,0\rangle$. What is $\vec{r}(\pi)$?

Solution:
Integrate

$$\vec{r}''(t) = \langle 1, 0, \sin(2t)\rangle$$

to get $\vec{r}'(t) = \langle t, 0, -\frac{\cos(2t)}{2}\rangle + \vec{C}$. To make this match with the initial velocity, $\vec{r}\,'(0) = \langle 1,0,0\rangle$, we have $\vec{C} = \langle 1, 0, 1/2\rangle$. Thus,

$$\vec{r}\,'(t) = \langle 1+t, 0, \frac{1}{2} - \frac{\cos(2t)}{2}\rangle.$$

Now integrate again:$\vec{r}(t) = \langle t + \frac{t^2}{2}, 0, \frac{t}{2} - \frac{\sin(2t)}{4}\rangle + \vec{C}$. To make this match with the initial position $\vec{r}(0) = \langle 1,1,2\rangle$, we have $\vec{C} = \langle 1,1,2\rangle$. Thus, the parametrization of the curve is

$$\vec{r}(t) = \langle 1 + t + \frac{t^2}{2}, 1, 2 + \frac{t}{2} - \frac{\sin(2t)}{4}\ \rangle.$$

At $t = \pi$ we have $\langle 1 + \pi + \frac{\pi^2}{2}, 1, 2 + \pi/2\rangle$.

Main definitions

A **parametrization** of a planar curve is a map $\vec{r}(t) = \langle x(t), y(t) \rangle$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. The parametrization of a space curve is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. The **image** of r is a **parametrized curve** in space.

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is called the **velocity** at time t. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ is called **unit tangent vector** or direction of motion. The vector $\vec{r}\,''(t)$ is called the **acceleration**. When knowing the acceleration and $\vec{r}\,'(0)$ and $\vec{r}(0)$ we can get back position $\vec{r}(t)$ by integration. Similarly, if we know $\vec{r}\,''(t)$ at all times and $\vec{r}(0)$ and $\vec{r}\,'(0)$, we can compute $\vec{r}(t)$ by integration.