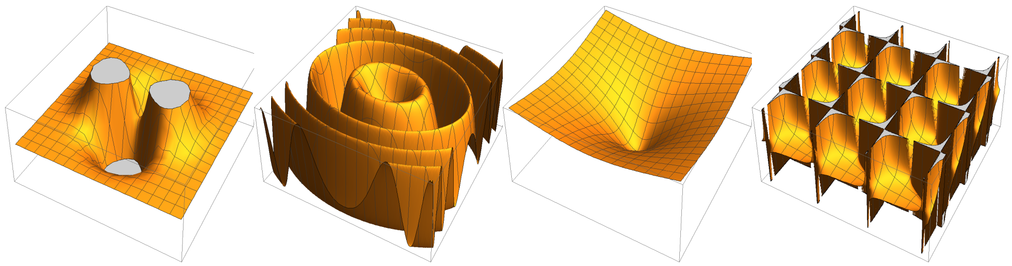
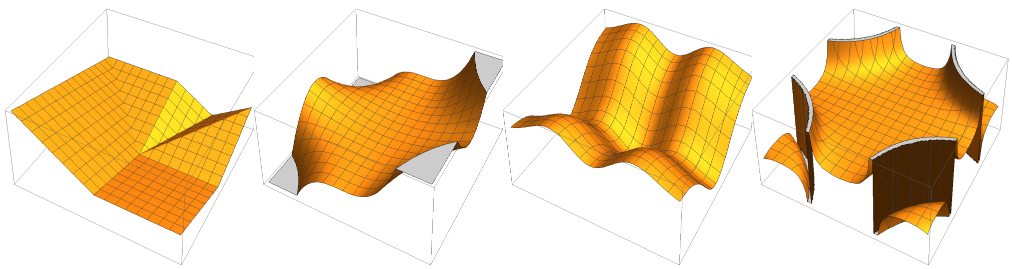
Homework 4: Functions of 2 and 3 variables

This homework is due Friday, 9/18 rsp Tuesday 9/22.

1 Match the following graphs with the functions $f(x,y)$.



I II III IV



V VI VII VIII

$f(x,y) =$	I,II,III,IV,V,VI,VII,VIII
$\lvert x - \lvert y - \lvert x \rvert \rvert \rvert$	
$x^2 + x^3 y^2$	
$\exp(-x^2 - y^2)(x^2 - y^2)$	
$\sin(x^2 + 2y^2)$	
$\log(x^2 + y^2 + 1)$	
$\exp(-x^2)x^2 - \exp(-y^2)$	
$\sec(xy)$	
$\tan(x)/\tan(y)$	

Solution:

(I) Graph I corresponds to $\exp(-x^2 - y^2)(x^2 - y^2)$. We can rewrite this as $\frac{x^2-y^2}{e^{x^2+y^2}}$. From here, we can see that, as the magnitudes of x and y go to infinity, the value of the function goes to 0. The only graph that exhibits this behavior is graph I.

(II) Graph II corresponds to $\sin(x^2+2y^2)$. If we consider a fixed $z = k$ trace, we get that $k = \sin(x^2 + 2y^2)$. This implies that $x^2 + 2y^2 = \arcsin(k)$. We can see that this is just the equation for an ellipse with radius $\sqrt{\arcsin(k)}$, centered at the origin, and stretched along the y-axis by a factor of $1/\sqrt{2}$. Thus, the traces are concentric ellipses. This narrows us down to graphs II and III. However, II is the one that exhibit sin-like behavior and this leads us to choose graph II.

(III) Graph III corresponds to $\log(x^2+y^2+1)$. Consider a fixed $z = k$ trace. The trace is $k = \log(x^2 + y^2 + 1)$. This implies that $e^k - 1 = x^2+y^2$. We see that the traces are circles centered at the origin. This leads us to either graph II or graph III. To decide between them, we note that as x and y go to infinity, $f(x, y)$ goes to infinity as well. Thus, we choose graph III, as graph II appears to be bounded.

(IV) Graph IV corresponds to $\tan(x)/\tan(y)$. Because $\tan(x)$ is a periodic function, we expect the graph to exhibit periodic behavior. This narrows us down to graphs II and IV. Looking at traces, when $z = k$, we get that $k = \frac{\tan(x)}{\tan(y)}$. Thus, $x = \arctan(k \tan(y))$. This is not elliptical, but it appears that graph II has elliptical traces when $z = k$. Therefore, the equation must correspond to graph IV.

Solution:
(V) Graph V corresponds to $|x - |y - |x|||$. The $x = 0$ trace is $f(0, y) = |y|$. The $y = 0$ trace is $f(x, 0) = 0$ if $x \geq 0$ and $f(x, 0) = 2x$ if $x < 0$. This behavior is exhibited by graph V.
(VI) Graph VI corresponds to $x^2 + x^3y^2$. We can see that, as $x \to \infty$, $f(x, y) \to \infty$ and as $x \to -\infty$, $f(x, y) \to -\infty$. Also, when $y = 0$, the trace is $f(x, 0) = x^2$. Combining this information, the only graph that fits this description is graph VI.
(VII) Graph VII corresponds to $\exp(-x^2)x^2 - \exp(-y^2)$. Consider the fixed-$y$ traces. When $y = k$, we get a trace of $f(x, k) = \frac{x^2}{e^{x^2}} - \frac{1}{e^{k^2}}$. This looks like two bumps that happen near the origin. As we change the value of k, we shift the trace up and down. At $k = 0$, we get $f(x, 0) = \frac{x^2}{e^{x^2}} - 1$. As k goes to $\pm\infty$, we get $f(x, k) = \frac{x^2}{e^{x^2}}$. The only graph that matches these traces is graph VII.
(VIII) Graph VIII corresponds to $\sec(xy)$. Along the lines $x = 0$ and $y = 0$, we get $f(x, y) = \sec(0) = 1$. As $\cos(xy) \to 0$, so $xy \to n * \frac{\pi}{2}$ for an integer n, our function behaves asymptotically, going to ∞ and $-\infty$ depending on the direction of the approach. These behaviors are exhibited in graph VIII.

2 a) Plot the graph of the function $z = x^4 e^{-x^4 - y^4}$. What are the traces of the graph, the intersection of the graph with the coordinate planes?
b) Find the domain and range of the function $f(x, y) = \log(\frac{x^2-1}{y^2-1})$ and plot the graph, where defined.

Solution:

a) If you look at traces, you see for $y = 0$ that the function $x^4 e^{-x^4}$ has two bumps. These bumps will become smaller. The graph looks like two long mountains arranged in a parallel way.

b) The function is defined where either both $|x| < 1, |y| < 1$ or $|x| > 1, |y| > 1$. This is the union of the unit square as well as 4 infinite regions like $x > 1, y > 1$ $x > 1, y < -1$, $x < 1, x > 1$ and $x < 1, y < 1$. The plot is difficult to do without help from technology.

3 a) Plot the graph and contour map of the function $f(x, y) = \frac{x-y}{x^2+y^2}$

b) Plot the graph and contour map of the function $f(x, y) = \frac{(x/y)}{x^2+y^2}$. You can explore this with technology.

Solution:

Also here, the plots are difficult to do without technology. In both cases, there is an obvious problem at $(0, 0)$.

a) One could plot the level curves in this case. Completion of square shows that each $f(x, y) = c$ is a circle.

b) Also this needs to be plotted with help. One can analyze it better in Polar coordinates, which we will cover next week.

4 Find an equation for the surface consisting of all points P for which the distance from P to the x-axes is twice the distance from P to the yz-plane. Identify the surface.

Solution:
The distance from a point to the x-axis is $\sqrt{y^2 + z^2}$. The distance to the yz-plane is $|x|$. The surface consisting of all these points is defined by $\sqrt{y^2 + z^2} = 2|x|$. This is a perfectly acceptable answer, but simplifying is nicer: square both sides and we get $y^2 + z^2 = 4x^2$. From this equation, we can determine that this is a cone whose axis is the x-axis.

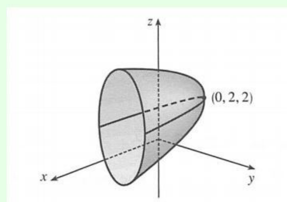
5 a) Draw the surface $4y^2 + z^2 - x - 16y - 4z + 20 = 0$.
b) Draw the surface $x - z^2 + y^2 + 4y = 1$.

Solution:

(a) Rearranging this equation and completing the square yields:

$$\begin{aligned} 4y^2 - 16y + z^2 - 4z &= x - 20 \\ 4(y^2 - 4y + 4) + z^2 - 4z + 4 &= x - 20 + 16 + 4 \\ 4(y-2)^2 + (z-2)^2 &= x \end{aligned}$$

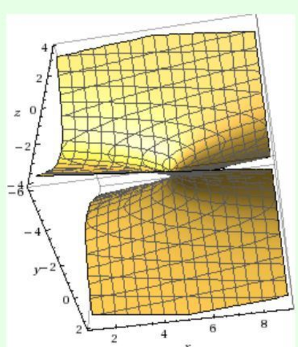
We can see that this is the equation of an elliptic paraboloid with vertex (0, 2, 2) that opens in the positive-x direction along the axis $y = 2, z = 2$. The paraboloid is stretched in the y direction by a factor of $\frac{1}{2}$.



(b) Rearranging this equation and completing the square yields:

$$\begin{aligned} x - z^2 + y^2 + 4y + 4 &= 5 \\ x - z^2 + (y+2)^2 &= 5 \\ x &= z^2 - (y+2)^2 + 5 \end{aligned}$$

We can see that this equation describes a hyperbolic paraboloid (i.e. a saddle). The saddle is centered at (0, -2, 0).



Main definitions

The **domain** D of a function $f(x,y)$ is the set of points where f is defined, the range is $\{f(x,y) \mid (x,y) \in D\}$. The **graph** of $f(x,y)$ is the surface $\{(x,y,f(x,y)) \mid (x,y) \in D\}$ in space. The set $f(x,y) = c = \text{const}$ is **contour curve** or **level curve** of f. The collection of all contour curves $\{f(x,y) = c\}$ is called the **contour map** of f. A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. We can visualize it by **contour surfaces** $g(x,y,z) = c$, where c is constant. Important are **traces**, the intersections of the surfaces with the coordinate planes.

The elliptic paraboloid $z-x^2-y^2 = 0$ and hyperboloid $z-x^2+y^2 = 0$ are examples of graphs $z-f(x,y) = 0$. The one sheeted hyperboloid $x^2+y^2-z^2 = 1$ and two sheeted hyperboloid $x^2+y^2-z^2 = -1$ or cylinder $x^2+y^2 = 1$ are examples of surfaces of revolution $x^2+y^2-g(z) = 0$.