Homework 31: Divergence Theorem

This last homework is due Wednesday, 12/2 rsp Thursday 12/3.

1 Find the flux of the field $\vec{F}(x, y, z) = \langle 2x^2 + 2z^{10}, 2xy + x, 8z - y \rangle$ through the boundary of the solid bounded by paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. by using the divergence theorem.

Solution:
$\text{div}\vec{F} = 4x + 2x + 2 = 6x + 8$. Use cylindrical coordinates to compute

$$\iiint_E \text{div}\vec{F} dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (6r\cos\theta + 8) r dz dr d\theta \ .$$

The integral

$$\int_0^2 \int_0^{2\pi} r(6r\cos\theta + 8)(4 - r^2) d\theta dr$$

simplifies to 64π.

2 Find the flux of the vector field $\vec{F}(x, y, z) = \langle x^2y + \cos^6(y), xy^2, 2xyz + e^{\sin(x)} \rangle$ through the outwards oriented solid bound by $x = 0, y = 0, z = 0$, and $x + 2y + z = 2$.

Solution:
The divergence of $\vec{F}$ is

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(xy^2)+\frac{\partial}{\partial z}(2xyz) = 2xy+2xy+2xy = 6xy.$$

Divergence Theorem turns the flux integral into a triple integral. Thus we get

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} &= \iiint_E 6xy\,dV = \int_0^1\int_0^{2-2y}\int_0^{2-x-2y} 6xy\,dz\,dx\,dy \\
&= \int_0^1\int_0^{2-2y} 6xy(2-x-2y)dx\,dy \\
&= \int_0^1\int_0^{2-2y}(12xy-6xy^2-12xy^2)\,dx\,dy \\
&= \int_0^1\left[6x^2y-2x^3y-6x^2y^2\right]_{x=0}^{2-2y}\,dy \\
&= \int_0^1 y(2-2y)^3\,dy = \left[-\frac{8}{5}y^5+6y^4-8y^3+4y^2\right]_0^1 \\
&= \frac{2}{5}.
\end{aligned}$$

3 Evaluate the flux $\int\int_S \vec{F}\cdot d\vec{S}$ where S is the hemisphere $z = \sqrt{1-x^2-y^2}$ together with the disk $x^2+y^2\leq 1$ in the xy plane and where $\vec{F}(x,y,z) = \langle x,y,z\rangle/\sqrt{x^2+y^2+z^2}$.

Solution:
To compute the divergence of $\vec{F}$, we begin by computing

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{|\vec{r}|}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) \\ &= \frac{\sqrt{x^2+y^2+z^2} - x^2/\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} \\ &= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} = \frac{|\vec{r}|^2 - x^2}{|\vec{r}|^3}\end{aligned}$$

Therefore,

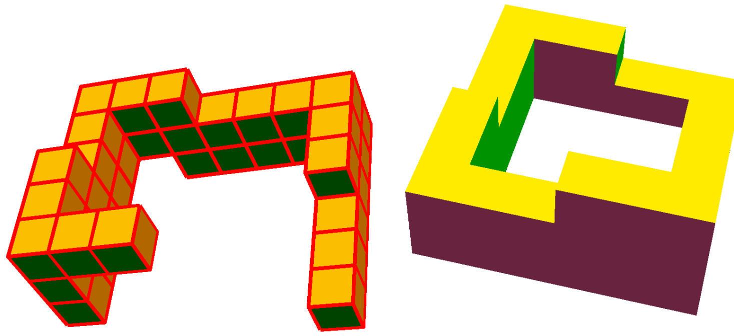
$$\begin{aligned}\mathrm{div}\vec{F} &= \frac{\partial}{\partial x}\left(\frac{x}{|\vec{r}|}\right) + \frac{\partial}{\partial y}\left(\frac{y}{|\vec{r}|}\right) + \frac{\partial}{\partial y}\left(\frac{y}{|\vec{r}|}\right) \\ &= \frac{|\vec{r}|^2 - x^2}{|\vec{r}|^3} + \frac{|\vec{r}|^2 - y^2}{|\vec{r}|^3} + \frac{|\vec{r}|^2 - z^2}{|\vec{r}|^3} \\ &= \frac{3|\vec{r}|^2 - (x^2+y^2+z^2)}{|\vec{r}|^3} = \frac{2|\vec{r}|^2}{|\vec{r}|^3} = \frac{2}{|\vec{r}|} \\ &= \frac{2}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

Using spherical coordinates to integrate over the solid hemisphere, the Divergence Theorem implies that:

$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{S} &= \iiint_E \frac{2}{\sqrt{x^2+y^2+z^2}}\, dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \frac{2}{\rho}\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\ &= 2\int_0^{\pi/2} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho\, d\rho = 2\cdot 1\cdot 2\pi\cdot\frac{1}{2} \\ &= 2\pi.\end{aligned}$$

4 Find $\int\int_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle x+\sin(y)+e^z, y+\sin(z)+e^z, z+\sin(x)+e^y\rangle$ and S is the boundary of the Escher stair solid displayed in the picture. The right picture shows the same figure from an other angle leading to the illusion. Each brick is a cube

of unit length 1.



Solution:

a) By the Divergence Theorem, $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_E \operatorname{div} \vec{F} dV = 3\,(\text{volume of } E) = 37 * 3$.

5 a) Use an integral theorem to evaluate $\int\int_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle x^2 yz, yz^2, z^3 e^{xy} \rangle$, where is the part of upwards oriented surface $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$.
b) Use an integral theorem to compute the line integral of $\vec{F}(x, y, z) = \langle x^3, y^5, 2z \rangle$ along the path $\vec{r}(t) = \langle \cos(t) + t^{100} \sin(17t), \sin(t) + \sin(20t), t \rangle$ from $t = 0$ to $t = 10\pi$.

Solution:
a) $\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$ where C:

$$\begin{aligned} \vec{r}(t) &= \langle 2\cos t, 2\sin t, 1\rangle, 0 \le t \le 2\pi \\ \vec{r}\,'(t) &= \langle -2\sin t, 2\cos t, 0\rangle \\ \vec{F}(\vec{r}(t)) &= \langle 8\cos^2 t \sin t, 2\sin t, e^{4\cos t \sin t}\rangle \\ \text{and } \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= -16\cos^2 t \sin^2 t + 4\sin t \cos t \end{aligned}$$

Thus $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\cos^2 t \sin^2 t + 4\sin t \cos t) dt = \left[-16\left(-\frac{1}{4}\sin t \cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi$
b) This is a gradient field with potential $x^4/4 + y^5/5 + z^2$. The initial point is $(1,0,0)$. The end point is $(1,0,10\pi)$. The result is $100\pi^2$.

Main points

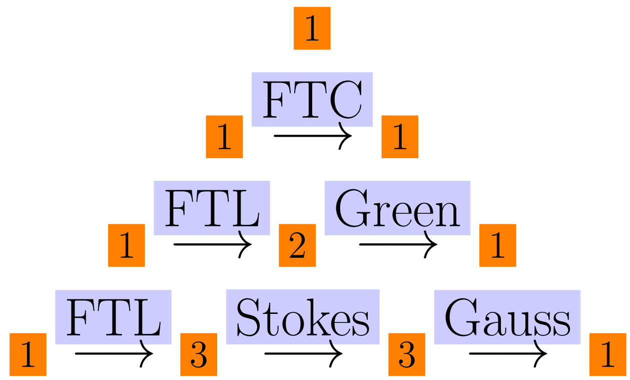
Divergence Theorem.

$$\int\int\int_E \text{div}(\vec{F})\, dV = \int\int_S \vec{F} \cdot dS \, .$$

All integral theorems are incarnations of **the fundamental theorem of multivariable Calculus**

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the **boundary** of G.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