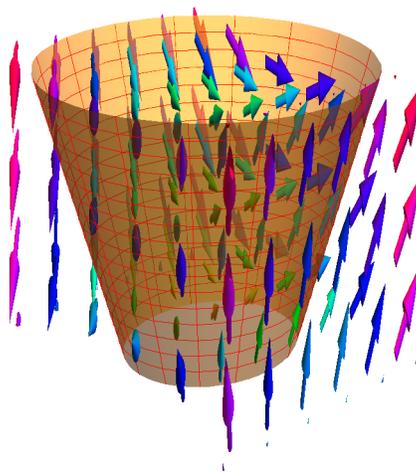


Homework 30: Stokes Theorem

This homework is due Monday, 11/30 or Tuesday 12/1 after thanksgiving. Do it before going into the break!

- 1 Evaluate $\int_S \text{curl}(\mathbf{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle x, y, z + (z-1)(z-2) \rangle$, where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. The cone can be parametrized by $\vec{r}(u, v) = \langle v \cos(u), v \sin(u), v \rangle$.



Solution:

The boundary consists of two circles. At $z = 1$ we have $\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$ at $z = 2$ we have $\vec{r}(t) = \langle \cos(t), -\sin(t), 2 \rangle$. Note that we have oriented the upper circle clockwise so that the surface is to the left. Both integrals are zero. The result is zero. We could also see it from the fact that vector field is a gradient field. Actually, since the curl is zero, the result is also obvious without Stokes. The integral of the zero field over the surface is zero.

- 2 Evaluate the line integral $\int_C \vec{F} \cdot d\mathbf{r}$, where $\vec{F}(x, y, z) = \langle e^{-x}, e^x, e^z \rangle$ and C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant, oriented counterclockwise as viewed from above.

Solution:

$\text{curl} \vec{F} = \langle 0, 0, e^x \rangle$. As S is the graph $f(x, y) = 1 - x - \frac{1}{2}y$ over

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x\}$$

we can parametrize it as $\vec{r}(x, y) = \langle x, y, 1 - x - \frac{1}{2}y \rangle$ for (x, y) in D . The normal vector

$$\vec{r}_x \times \vec{r}_y = \langle 1, \frac{1}{2}, 1 \rangle.$$

is upward orientation of S , which is compatible with the given orientation of C . Stokes's Theorem gives

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \iint_D \langle 0, 0, e^x \rangle \cdot (\vec{r}_x \times \vec{r}_y) \, dx \, dy \\ &= \iint_D \langle 0, 0, e^x \rangle \cdot \langle 1, \frac{1}{2}, 1 \rangle \, dx \, dy \\ &= \int_0^1 \int_0^{2-2x} e^x \, dy \, dx = \int_0^1 (2 - 2x)e^x \, dx \\ &= 2e - 4. \end{aligned}$$

- 3 Evaluate the line integral $\int_C \vec{F} \cdot d\mathbf{r}$, where $\vec{F}(x, y, z) = \langle xy, 2z, 3y \rangle$ and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

Solution:

The intersection of the plane and the cylinder is an ellipse lying over the disk D given by $x^2 + y^2 \leq 9$ in the xy -plane. We parametrize by $\vec{r}(x, y) = \langle x, y, 5 - x \rangle$ for (x, y) lying the ellipse D . The usual computation finds

$$\vec{r}_x \times \vec{r}_y = \langle 1, 0, 1 \rangle \text{ and } \text{curl} \vec{F} = \langle 1, 0, -x \rangle.$$

(Notice that the upward orientation from our parametrization of S is compatible with the orientation of C .) Now we apply Stokes's Theorem, eventually switching to polar coordinates to compute the integral:

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_D \text{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) \, dx \, dy \\ &= \iint_D \langle 1, 0, -x \rangle \cdot \langle 1, 0, 1 \rangle \, dx \, dy = \iint_D (1 - x) \, dx \, dy \\ &= \int_0^{2\pi} \int_0^3 (1 - r \cos \theta) r \, dr \, d\theta = \int_0^{2\pi} \left(\frac{9}{2} - 9 \cos \theta \right) \, d\theta \\ &= 9\pi. \end{aligned}$$

- 4 Compute both sides of Stokes' Theorem for $\vec{F}(x, y, z) = \langle -2yz, y, 3x \rangle$ and the surface S which is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upwards.

Solution:

The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4, z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 1 \rangle, 0 \leq t \leq 2\pi$, and then $\vec{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle$. Thus $\vec{F}(\vec{r}(t)) = \langle -4 \sin t, 2 \sin t, 6 \cos t \rangle, \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 8 \sin^2 t, 4 \sin t \cos t, 0 \rangle$ and

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (8 \sin^2 t + 4 \cos t \sin t) dt \\ &= 8 \left(\frac{1}{2} t - \frac{1}{4} \sin 2t \right) + 2 \sin^2 t \Big|_0^{2\pi} \\ &= 8\pi \end{aligned}$$

Now $\text{curl } \vec{F} = \langle 0, -3 - 2y, 2z \rangle$, and the projection D of S on the xy -plane is the disk $x^2 + y^2 \leq 4$, so by Equation 13.6.10 with $z = g(x, y) = 5 - x^2 - y^2$ we have

$$\begin{aligned} \iint_S \text{curl } \vec{F} \cdot d\vec{S} &= \iint_D [-0 - (-3 - 2y)(-2y) + 2z] dA \\ &= \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2)] dA \\ &= \int_0^{2\pi} \int_0^2 [-6r \sin \theta - 4r^2 \sin^2 \theta + 2(5 - r^2)] r dr d\theta \\ &= \left[-2r^3 \sin \theta - r^4 \sin^2 \theta + 5r^2 - \frac{1}{2}r^4 \right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} (-16 \sin \theta - 16 \sin^2 \theta + 20 - 8) d\theta \\ &= 16 \cos \theta - 16 \left(\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right) + 12\theta \Big|_0^{2\pi} \\ &= 8\pi \end{aligned}$$

- 5 a) Evaluate $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$ with $\vec{F}(x, y, z) = \langle y + \sin x, z^2 + \cos y, x^3 \rangle$, where C is the curve $\vec{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle, 0 \leq t \leq 2\pi$ which as you can see lies on the surface $z = 2xy$.

b) Explain without doing any computation that if S is the torus $\vec{r}(u, v) = \langle (2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(u) \rangle$ with $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$ and \vec{F} is a vector field like $\vec{F}(x, y, z) = \langle e^{e^x}, \sin \sin(y + z + x), x^{100} \rangle$ then $\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Solution:

(a) The curl is $\vec{F} = \langle -2z, -3x^2, -1 \rangle$. Since $\sin 2t = 2 \sin t \cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C . Then the projection of S onto the xy -plane is the unit disk $D[x^2 + y^2 \leq 1]$. C is traversed clockwise (when viewed from above so S is oriented downward). Using Equation 13.6.10 with $g(x, y) = 2xy$, $P = -2z = -4xy$, $Q = -3x^2$, $R = -1$ and multiplying by -1 for the downward orientation, we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= -\iint_S \text{curl} \vec{F} \cdot d\vec{S} = -\iint_D [-(-4xy)(2y) - (-3x^2)(2x)] dA \\ &= -\iint_D (8xy^2 + 6x^3 - 1) dA \\ &= -\int_0^{2\pi} \int_0^1 (8r^3 \cos \theta \sin^2 \theta + 6r^3 \cos^3 \theta - 1) r dr d\theta \\ &= -\int_0^{2\pi} \left(\frac{8}{5} \cos \theta \sin^2 \theta + \frac{6}{5} \cos^3 \theta - \frac{1}{2} \right) d\theta \\ &= -\left[\frac{8}{15} \sin^3 \theta + \frac{6}{5} \left(\sin \theta - \frac{1}{3} \sin^3 \theta \right) - \frac{1}{2} \sin^3 \theta - \frac{1}{2} \theta \right]_0^{2\pi} \\ &= \pi \end{aligned}$$

6 Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S . Then $\iint_S \text{curl} \vec{F} \cdot d\vec{S} = \iint_{H_1} \text{curl} \vec{F} \cdot d\vec{S} + \iint_{H_2} \text{curl} \vec{F} \cdot d\vec{S} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \vec{F} \cdot d\vec{r} = -\oint_{C_1} \vec{F} \cdot d\vec{r}$ so $\iint_S \text{curl} \vec{F} \cdot d\vec{S} = 0$.

Main points

Stokes theorem: Let S be a surface bounded by a curve C and \vec{F} be a vector field. Then

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$

The orientation of S is given by the parametrization: the orientation of C is such that if you walk along C with the head in the "up" direction $\vec{r}_u \times \vec{r}_v$ and your nose into the \vec{r}' direction, then your left foot is on the surface.

Written out in detail, we have

$$\int \int_R \text{curl}(\vec{F}(\vec{r}(u, v))) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

A lot of things come together here: surfaces, curves, dot product, cross product, triple scalar product, vector fields, double integrals and curl. What does it mean? From a SMBC cartoon: **"Stokes theorem? Yeah, thats how if you draw a loop around something, you can tell how much swirly is in it."**