Homework 3: Cross product, lines, planes

This homework is due Wednesday, 9/16 rsp Thursday 9/17.

1 a) Find a nonzero vector orthogonal to the plane through the points $P = (-2, 3, 1), Q = (1, 5, 2), R = (4, 3, -1)$ and containing P.
b) Find the equation of this plane. c) Find the area of the triangle PQR.

Solution:
a) Two vectors in the plane through the points P, Q, and R are the vectors $\overrightarrow{PQ} = \langle 3, 2, 1 \rangle$ and $\overrightarrow{PR} = \langle 6, 0, -2 \rangle$. Thus, a vector orthogonal to the plane through P, Q, R is
$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -4, 12, -12 \rangle$ (or any scalar multiple thereof).
c) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -4, 12, -12 \rangle| = \sqrt{16 + 144 + 144} = \sqrt{304}$. Thus, the area of triangle PQR is $\frac{1}{2}\sqrt{304}$.

2 a) Parametrize a line perpendicular to the plane containing $A = (1, 1, 1), B = (2, 3, 4)$ and $C = (4, 5, 6)$ and passing through A.
b) Find the equation $ax + by + cz = d$ of the plane through A, B, C.

Solution:
a) The vector $\vec{AB} \times \vec{AC} = \langle 1, 2, 3 \rangle \times \langle 3, 4, 5 \rangle = \langle -2, 4, -2 \rangle$ is normal to the plane. The parametrization is $\vec{r}(t) = \langle 1, 1, 1 \rangle + t\langle -2, 4, -2 \rangle$.
b) The plane is is obtained by plugging in a point, like $A = (1, 1, 1)$. The equation is $-2x + 4y - 2z = 0$.

3 a) Use volume to determine whether $A = (1, 1, 2)$, $B = (3, -1, 6)$, $C = (5, 2, 0)$ and $D = (1, -4, 12)$ are in the same plane.
b) Find the distance between the line L through A, B and the line M through C, D.
c) How come that whenever A, B, C, D are not in the same plane, then the distance between L and M is positive?

Solution:

a) Begin by defining the vectors $\vec{u} = \overrightarrow{AB} = \langle 2, -2, 4\rangle, \vec{v} = \overrightarrow{AC} = \langle 4, 1, -2\rangle$ and $\vec{w} = \overrightarrow{AD} = \langle 0, -5, 10\rangle$. These three vectors determine a parallelopiped whose volume can be computed by their triple scalar product. If the vectors are coplanar, then the volume of this parallelopiped will be 0. Compute the triple scalar product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle 2, -2, 4\rangle \cdot \langle 0, -40, -20\rangle = 0 + 80 - 80 = 0.$$

Thus, the volume of the parallelepiped is 0, so the vectors are coplanar and so the points A, B, C and D that define the vectors lie in the same plane.

b) Since the distance is the volume divided by the area $|\vec{v} \times \vec{w}|$, also the distance is zero.

c) The four points are in the same plane if and only the volume of the parallel epiped is positive. But this is equivalent to the distance being positive.

4 a) Find an equation of the plane containing the line of intersection of the planes $x - z = 1$ and $y + z = 3$ which is perpendicular to the plane $x + y - 2z = 1$.

b) Find the distance of the plane found in a) to the origin $(0, 0, 0)$.

Solution:

a) To find the equation of a plane, we need to find a point and two vectors (that are not multiples of one another) in the plane. Since our plane is perpendicular to $x + y - 2z = 1$, it must contain $\vec{d_1} = \langle 1, 1, -2\rangle$ as a tangent vector. To find another vector in our plane, we note that the direction vector of the line of intersection also lies in our plane. This direction vector lies in both the plane $x - z = 1$ and the plane $y + z = 3$, so it is perpendicular to both their normal vectors. So, we compute:

$$\vec{d_2} = \langle 1, 0, -1\rangle \times \langle 0, 1, 1\rangle = \langle 1, -1, 1\rangle.$$

Thus, the plane we are looking for has two tangent vectors: $\langle 1, 1, -2\rangle$ and $\langle 1, -1, 1\rangle$. To find the normal vector to our plane, we cross these two to get

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \langle 1, 1, -2\rangle \times \langle 1, -1, 1\rangle = \langle -1, -3, -2\rangle.$$

This gives us the equation

$$-x - 3y - 2z = d.$$

Since the point (1, 3, 0) lies in our plane (it lies on the line of intersection), we use it to find that $d = -10$. Thus the equation our plane is:
$x + 3y + 2z = 10$.

b) We use the formula for the distance from a point P to a plane Σ: $d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$. Here, $\vec{n}$ is the normal vector to the plane, so $\vec{n} = \langle 1, 3, 2\rangle$. We have $P = (0, 0, 0)$ as the point away from the plane and $Q = (1, 3, 0)$ as the point on the plane. Thus $\vec{PQ} = \langle 1, 3, 0\rangle$ and the distance from our plane to the origin is

$$\frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|\langle 1, 3, 0\rangle \cdot \langle 1, 3, 2\rangle|}{|\langle 1, 3, 2\rangle} = \frac{10}{\sqrt{1 + 9 + 4}} = \frac{10}{\sqrt{14}}.$$

5 a) Parametrize the line L through $P = (2, 1, 2)$ that intersects the line $x = 1 + t, y = 1 - t, z = 2t$ perpendicularly.

b) What is the distance from this line L to the origin $(0, 0, 0)$?

Solution:

a) The point $(1,1,0)$ is on the line. The vector $\vec{v} = \langle 1,-1,2 \rangle$ is inside the line. If $X = (1+t, 1-t, 2t)$ is a general point on the line, then $\vec{PX} = \langle t-1, -t, 2t-2 \rangle$ must be perpendicular to $\langle 1,-1,2 \rangle$. The dot product is $\vec{PX} \cdot \vec{v} = \langle t - 1 + t + 4t - 4 = 6t - 5 = 0$ gives $t = 5/6$ and the point $X = (7/6, 5/6, 2/6)$ and vector $\vec{PX} = \langle -1/6, -5/6, -1/3 \rangle$ which is parallel to $\langle 1,5,2 \rangle$. The equation of the line is $\vec{r}(t) = \langle 1,1,0 \rangle + t\langle 1,5,2 \rangle$.

b) Use the distance formula and compute $|\vec{PO} \times \langle 1,5,2 \rangle|\sqrt{149}/\sqrt{30} = |\langle 8,2,-9 \rangle|/|\langle 1,5,2| = \sqrt{149/30}$.

Main definitions

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle \ .$$

The number $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. It satisfies $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$. The **orientation** given by the sign of $[\vec{u}, \vec{v}, \vec{w}]$.

A point $P = (p, q, r)$ and a vector $\vec{v} = \langle a, b, c \rangle$ define the **line** $L = \{ \langle x, y, z \rangle = \langle p, q, r \rangle + t \langle a, b, c \rangle, t \in \mathbb{R} \ \}$.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{ \vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.
An example is $\Sigma = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t \langle 2, 4, 6 \rangle + s \langle 1, 0, -1 \rangle \ \}$. This is called the **parametric description** of a plane. The implicit equation of the plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$ is $ax + by + cz = d$, where $\langle a, b, c \rangle = \vec{v} \times \vec{w}$ is a vector normal to the plane and d is obtained by plugging in $\vec{x}_0$.