Homework 29: Flux integral, Stokes I

This homework is due Monday, 11/23 rsp Tuesday 11/24 just before Thanksgiving. If no orientation is given, the orientation is assumed to be "outwards". It is no problem if Stokes theorem has not been covered in your section. There is one problem which gives a first exposure to this important theorem. The Monday/Tuesday lecture before thanksgiving will cover it in more detail.

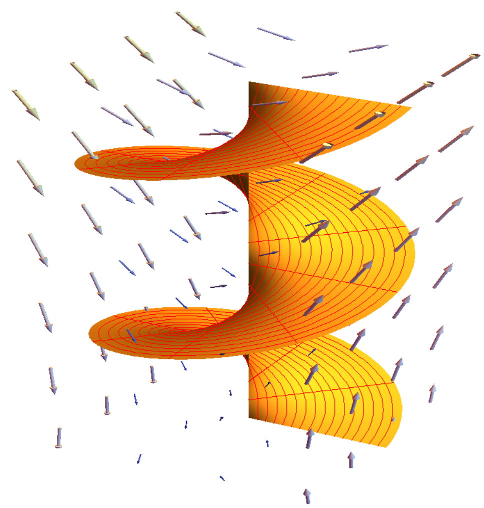
1 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ if

$$\vec{F}(x, y, z) = \langle 3z, 3y, 3x \rangle ,$$

and S is the helicoid

$$\vec{r}(u, v) = \langle u\cos v, u\sin v, v \rangle, 0 \le u \le 1, 0 \le v \le 4\pi$$

which has an upward orientation.



2 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle x, y, 5 \rangle ,$$

where S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$.

3 The temperature $f(x, y, z)$ at a point (x, y, z) is equal to the distance from the center $(0, 0, 0)$. Find the flux of the heat flow field $\vec{F} = -\nabla f$ across a sphere S of radius 2 centered at $(0, 0, 0)$.

4 Let $\vec{F}(x, y, z)$ be an inverse square field, that is

$$\vec{F}(x, y, z) = c\langle x, y, z\rangle/\rho^3$$

with $\rho = \sqrt{x^2 + y^2 + z^2}$. Show that the flux of $\vec{F}$ across a sphere S with center at the origin and radius R is independent of the radius of S.

5 Use Stokes theorem to evaluate the flux integral $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle xz, x, y\rangle \ ,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 25, z \geq 0$, oriented upwards. Stokes theorem expresses this as a line integral along the boundary curve $\vec{r}(t) = \langle 5\cos(t), 5\sin(t), 0\rangle$, $0 \leq t \leq 2\pi$.

Main points

If a surface S is parametrized as $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ over a domain G in the uv-plane and $\vec{F}$ is a vector field, then the **flux integral** of $\vec{F}$ through S is

$$\int\int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \ dudv \ .$$

If $d\vec{S} = (\vec{r}_u \times \vec{r}_v) \ dudv$ represents an infinitesimal normal vector to the surface, this can be written as $\int\int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if $\vec{F} =$ fluid velocity field, then $\int\int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through S in unit time.

Stokes theorem tells that if S be a surface bounded by a curve C and $\vec{F}$ be a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \ \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \ .$$