

Homework 29: Flux integral, Stokes I

This homework is due Monday, 11/23 resp Tuesday 11/24 just before Thanksgiving. If no orientation is given, the orientation is assumed to be "outwards". It is no problem if Stokes theorem has not been covered in your section. There is one problem which gives a first exposure to this important theorem. The Monday/Tuesday lecture before thanksgiving will cover it in more detail.

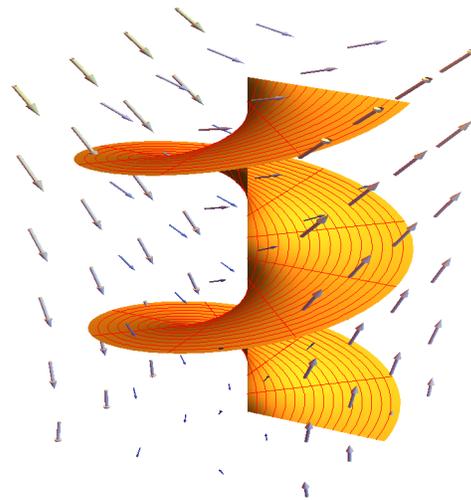
1 Evaluate the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ if

$$\vec{F}(x, y, z) = \langle 3z, 3y, 3x \rangle ,$$

and S is the helicoid

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, 0 \leq u \leq 1, 0 \leq v \leq 4\pi$$

which has an upward orientation.



Solution:

From the parameterization $\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$, we compute

$$\vec{r}_u = 3\langle \cos v, \sin v, 0 \rangle$$

$$\vec{r}_v = 3\langle -u \sin v, u \cos v, 1 \rangle.$$

$$\text{Thus } \vec{r}_u \times \vec{r}_v = 9\langle \sin v, -\cos v, u \rangle.$$

Since $\vec{F}(\vec{r}(u, v)) = \langle 3v, 3u \sin v, 3u \cos v \rangle$, we get

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \int_0^1 \int_0^{4\pi} \langle 3v, 3u \sin v, 3u \cos v \rangle \cdot \langle 3 \sin v, -3 \cos v, 3u \rangle du dv \\ &= 27 \int_0^1 \int_0^{4\pi} (v \sin v - u \sin v \cos v + u^2 \cos v) du dv. \end{aligned}$$

When we integrate v from 0 to 4π , the last two terms are 0, while $3 \int v \sin v dv = -12\pi$. Hence, $\iint_S \vec{F} \cdot d\vec{S} = -12\pi$.

2 Evaluate the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle x, y, 5 \rangle,$$

where S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$.

Solution:

Here S consists of three surfaces: S_1 , the lateral surface of the cylinder; S_2 , the front formed by the plane $x + y = 2$; and the back, S_3 , in the plane $y = 0$.

On S_1 : $\vec{F}(\vec{r}(\theta, y)) = \langle \sin \theta, y, 5 \rangle$ and $\vec{r}_\theta \times \vec{r}_y = \langle \sin \theta, 0, \cos \theta \rangle \Rightarrow$

$$\begin{aligned}\iint_{S_1} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^{2-\sin\theta} (\sin^2 \theta + 5 \cos \theta) dy d\theta \\ &= \int_0^{2\pi} (2 \sin^2 \theta + 10 \cos \theta - \sin^3 \theta - 5 \sin \theta \cos \theta) d\theta \\ &= 2\pi\end{aligned}$$

On S_2 : $\vec{F}(\vec{r}(x, z)) = \langle x, 2 - x, 5 \rangle$ and $\vec{r}_z \times \vec{r}_x = \langle 1, 1, 0 \rangle \Rightarrow$

$$\begin{aligned}\iint_{S_2} \vec{F} \cdot d\vec{S} &= \iint_{x^2+z^2 \leq 1} [x + (2 - x)] dA \\ &= 2\pi\end{aligned}$$

On S_3 : $\vec{F}(\vec{r}(x, z)) = \langle x, 0, 5 \rangle$ and $\vec{r}_x \times \vec{r}_z = \langle 0, -1, 0 \rangle \Rightarrow$

$$\iint_{S_3} \vec{F} \cdot d\vec{S} = 0$$

Hence $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$.

- 3 The temperature $f(x, y, z)$ at a point (x, y, z) is equal to the distance from the center $(0, 0, 0)$. Find the flux of the heat flow field $\vec{F} = -\nabla f$ across a sphere S of radius 2 centered at $(0, 0, 0)$.

Solution:

$u(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Then,

$$\vec{F} = -\nabla u = \left\langle -\frac{x}{\sqrt{x^2 + y^2 + z^2}}, -\frac{y}{\sqrt{x^2 + y^2 + z^2}}, -\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$$

The outward unit normal to the unit sphere of radius a is $\vec{n} = \langle x/a, y/a, z/a \rangle$. Therefore

$$\vec{F} \cdot \vec{n} = \frac{\langle -x, -y, -z \rangle \cdot \langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2 + z^2}} = -1.$$

Thus, the rate of heat flow across S is

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \vec{n} \, dS \\ &= \iint_S -1 \, dS \\ &= -(\text{Surface area of } S) \\ &= -4\pi r^2 \\ &= -16\pi. \end{aligned}$$

- 4 Let $\vec{F}(x, y, z)$ be an inverse square field, that is

$$\vec{F}(x, y, z) = c\langle x, y, z \rangle / \rho^3$$

with $\rho = \sqrt{x^2 + y^2 + z^2}$. Show that the flux of \vec{F} across a sphere S with center at the origin and radius R is independent of the radius of S .

Solution:

Let $\vec{r} = \langle x, y, z \rangle$ and S be a sphere of radius ρ centered at the origin. Then $|\vec{r}| = \rho$ and $\vec{F}(\vec{r}) = c\vec{r}/|\vec{r}^3| = \left(\frac{c}{\rho^3}\right) \langle x, y, z \rangle$. A parametric representation for S is $\vec{r}(\phi, \theta) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$. Then $\vec{r}_\phi = \langle \rho \cos \phi \cos \theta, \rho \cos \phi \sin \theta, -\rho \sin \phi \rangle$, $\vec{r}_\theta = \langle -\rho \sin \phi \sin \theta, \rho \sin \phi \cos \theta, 0 \rangle$ and the outward orientation is given by

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \rho^2 \sin^2 \phi \cos \theta, \rho^2 \sin^2 \phi \sin \theta, \rho^2 \sin \phi \cos \phi \rangle$$

The flux of \vec{F} across S is

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \int_0^\pi \int_0^{2\pi} \frac{c}{\rho^3} \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle \\ &\quad \cdot \langle \rho^2 \sin^2 \phi \cos \theta, \rho^2 \sin^2 \phi \sin \theta, \rho^2 \sin \phi \cos \phi \rangle d\theta d\phi \\ &= \frac{c}{\rho^3} \int_0^\pi \int_0^{2\pi} \rho^3 (\sin^3 \phi + \sin \phi \cos^2 \phi) d\theta d\phi \\ &= c \int_0^\pi \int_0^{2\pi} \sin \phi d\theta d\phi \\ &= 4\pi c. \end{aligned}$$

Thus the flux does not depend on the radius ρ .

- 5 Use Stokes theorem to evaluate the flux integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle xz, x, y \rangle,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 25, z \geq 0$, oriented upwards. Stokes theorem expresses this as a line integral along the boundary curve $\vec{r}(t) = \langle 5 \cos(t), 5 \sin(t), 0 \rangle$, $0 \leq t \leq 2\pi$.

Solution:

Compute the line integral of \vec{F} along the circle of radius 5 in the $x - y$ plane.

Main points

If a surface S is parametrized as $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over a domain G in the uv -plane and \vec{F} is a vector field, then the **flux integral** of \vec{F} through S is

$$\int \int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv .$$

If $d\vec{S} = (\vec{r}_u \times \vec{r}_v) \, dudv$ represents an infinitesimal normal vector to the surface, this can be written as $\int \int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if $\vec{F} =$ fluid velocity field, then $\int \int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through S in unit time.

Stokes theorem tells that if S be a surface bounded by a curve C and \vec{F} be a vector field, then

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$