

Homework 28: Curl and Div

This homework is due Friday, 11/20 resp Tuesday 11/24.

- 1 Find a) the curl and b) the divergence of the vector field

$$\vec{F}(x, y, z) = \langle \sin(yz), \sin(zx), \sin(xy) \rangle .$$

Solution:

(a)

$$\begin{aligned} \operatorname{curl} \vec{F} &= \nabla \times \vec{F} \\ &= (x \cos xy - x \cos zx) \vec{i} - (y \cos xy - y \cos yz) \vec{j} \\ &\quad + (z \cos zx - z \cos yz) \vec{k} \\ &= x(\cos xy - \cos zx) \vec{i} - y(\cos xy - \cos yz) \vec{j} \\ &\quad + z(\cos zx - \cos yz) \vec{k} \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} \\ &= \frac{\partial}{\partial x}(\sin yz) + \frac{\partial}{\partial y}(\sin zx) + \frac{\partial}{\partial z}(\sin xy) \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

- 2 Let f be a scalar field and \vec{F} a vector field in space. State whether each expression is meaningful. If not, explain why. If so, state

whether it is a scalar field or a vector field. In all problems, we deal with functions and vector fields in 3D space.

- a) $\text{grad}(f) \times \text{div}(\vec{F})$
- b) $\text{div}(\text{curl}(\text{grad } f))$
- c) $\text{div}(\vec{F})$
- d) $\text{curl}(\text{grad}(f))$
- e) $\text{grad}(\vec{F})$
- f) $\text{grad}(\text{div}(\vec{F}))$
- g) $\text{div}(\text{grad}(f))$
- h) $\text{grad}(\text{div}(f))$
- i) $\text{curl}(\text{curl}(\vec{F}))$
- j) $\text{div}(\text{div}(\vec{F}))$
- k) $\text{curl}(f)$
- l) $\text{grad}(f)$

Solution:

- (a) $\text{grad}(f) \times \text{div}(\vec{F})$ is meaningless because $\text{div}\vec{F}$ is a scalar field.
- (b) $\text{div}(\text{curl}(\text{grad } f))$ is a scalar field.
- (c) $\text{div}\vec{F}$ is a scalar field.
- (d) $\text{curl}(\text{grad}(f))$ is a vector field.
- (e) $\text{grad}(\vec{F})$ is meaningless because \vec{F} is not a scalar field.
- (f) $\text{grad}(\text{div}(\vec{F}))$ is a vector field.
- (g) $\text{div}(\text{grad}(f))$ is a scalar field.
- (h) $\text{grad}(\text{div}(f))$ is meaningless because f is a scalar field.
- (i) $\text{curl}(\text{curl}(\vec{F}))$ is a vector field.
- (j) $\text{div}(\text{div}(\vec{F}))$ is meaningless because $\text{div}\vec{F}$ is a scalar field.
- (k) $\text{curl } f = \nabla \times f$ is meaningless because f is a scalar field.
- (l) $\text{grad } f$ is a vector field.

3 a) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = \langle 1, 2, 3 \rangle$$

If yes, find one.

b) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = \langle xyz, -y^2z, yz^2 \rangle ?$$

If yes, find one.

c) Assume \vec{F} is a gradient field. Does this imply that there is a vector field \vec{G} such that $\text{curl}(\vec{G}) = \vec{F}$? If yes, show it. If no, find a counter example.

Solution:

a) Yes, for example $\langle -3y, -z, -2x \rangle$. b) No. Assume there is such a G . Then $\text{div}(\text{curl}\vec{G}) = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(yz^2) = yz - 2yz + 2yz = yz \neq 0$ which contradicts Theorem 11.

c) We necessarily need the divergence to be zero. Take $\langle x, 0, 0 \rangle$ for example.

4 a) Verify that any vector field of the form

$$\vec{F}(x, y, z) = \langle f(x), g(y), h(z) \rangle$$

is irrotational.

b) Verify that any vector field of the form

$$\vec{F}(x, y, z) = \langle f(y, z), g(x, z), h(x, y) \rangle$$

is incompressible.

c) Find a nonzero vector field \vec{F} such that $\text{curl}(\vec{F}) = \langle 0, 0, 0 \rangle$.

Solution:

$$(a) \operatorname{curl} \vec{F} = \left(\frac{\partial h(z)}{\partial y} - \frac{\partial g(y)}{\partial z} \right) \vec{i} + \left(\frac{\partial f(x)}{\partial z} - \frac{\partial h(z)}{\partial x} \right) \vec{j} + \left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y} \right) \vec{k} = (0 - 0)\vec{i} + (0 - 0)\vec{j} + (0 - 0)\vec{k} = 0.$$

Hence \vec{F} is irrotational.

$$(b) \operatorname{div} \vec{F} = \frac{\partial (f(y, z))}{\partial x} + \frac{\partial (g(x, z))}{\partial y} + \frac{\partial (h(x, y))}{\partial z} = 0 \text{ so } \vec{F} \text{ is incompressible.}$$

(c) Take a gradient field.

5 a) Prove the identity $\operatorname{div}(\nabla f \times \nabla g) = 0$.

b) Show that every scalar function $f(x, y, z)$ is the divergence of some vector field \vec{F} .

Solution:

a) Let ∇f be $\langle P_1, Q_1, R_1 \rangle$ and $\nabla g = \langle P_2, Q_2, R_2 \rangle$. Then

$$\begin{aligned} \operatorname{div}(\nabla f \times \nabla g) &= \nabla \cdot (\nabla f \times \nabla g) \\ &= \frac{\partial}{\partial x}(Q_1 R_2 - R_1 Q_2) - \frac{\partial}{\partial y}(P_1 R_2 - R_1 P_2) + \frac{\partial}{\partial z}(P_1 Q_2 - Q_1 P_2) \\ &= \left[Q_1 \frac{\partial R_2}{\partial x} + R_2 \frac{\partial Q_1}{\partial x} - Q_2 \frac{\partial R_1}{\partial x} - R_1 \frac{\partial Q_2}{\partial x} \right] - \\ &\quad \left[P_1 \frac{\partial R_2}{\partial y} + R_2 \frac{\partial P_1}{\partial y} - P_2 \frac{\partial R_1}{\partial y} - R_1 \frac{\partial P_2}{\partial y} \right] + \\ &\quad \left[P_1 \frac{\partial Q_2}{\partial z} + Q_2 \frac{\partial P_1}{\partial z} - P_2 \frac{\partial Q_1}{\partial z} - Q_1 \frac{\partial P_2}{\partial z} \right] \\ &= \left[P_2 \left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z} \right) + Q_2 \left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x} \right) + R_2 \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right) \right] - \\ &\quad \left[P_1 \left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z} \right) + Q_1 \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x} \right) + R_1 \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y} \right) \right] \\ &= \nabla g \cdot \operatorname{curl}(\nabla f) - \nabla f \cdot \operatorname{curl}(\nabla g) \\ &= 0 \text{ (by Theorem 3)} \end{aligned}$$

b) This is very easy if we make the problem harder by assuming $P = 0, Q = 0$. Then $\operatorname{div}(F) = R_z$. The question is now whether any function can appear in this form. Yes, just take $R(x, y, z) = \int_0^z f(x, y, t) dt$.

Main points

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is the vector field $\operatorname{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.

The curl measure rotation of a field. If \vec{F} has zero curl everywhere it is **irrotational**. Remember, the curl of $\vec{F} = \langle P, Q \rangle$ is a scalar.

The **divergence** of a vector field $\vec{F}(x, y, z) = \langle P, Q, R \rangle$ is

$$\operatorname{div}(\vec{F})(x, y, z) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z).$$

$\operatorname{div}(\vec{F})$ measures the expansion of a field. Zero divergence everywhere is called **incompressible**.

$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$ for all vector fields \vec{F} . $\operatorname{curl}(\nabla f) = 0$ for all functions f .