Homework 27: Greens theorem

This homework is due Wednesday, 11/18 rsp Thursday 11/19.

1 Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \ ,$$

where $\vec{F}(x,y) = \langle 6y + x^5 - 1, \sin(y)y^{100} \rangle$ and C consists of the line segments from $(0,1)$ to $(0,0)$ and from (0,0) to (1,0) and the parabola $y = 1 - x^2$ from $(1,0)$ to $(0,1)$.

Solution:
We have $Q_x - P_y = -6$ so that the result is the area of the region below the parabola on the interval $[0,1]$. The result is -4.

2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$\vec{F}(x,y) = \langle y^2 \ \cos x + \sin(\sin(x)), 15x + x^2 + 2y \ \sin x + \sin(\sin(y)) \rangle$

where C is the triangular path from (0,0) to (2,6) to (2,0) to (0,0). Watch the orientation of the curve!
b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F}(x,y) = \langle y - \ln(x^2 + y^2), 2\tan^{-1}(y/x) \rangle$$

C is the circle $(x-2)^2 + (y-3)^2 = 16$ oriented counterclockwise. Don't worry here about the singularity or the fact that arctan is not continuous.

Solution:
a) The curl is 5. The area of the triangle is 6. The result is 90.
b) The curl is -1. The result is the area of the disk times -1 which is -16π. P.S. One can interpret the field so that is no problem with the singularity at $(0,0)$. Just in case you would be interested: the result can depend on the choice of the arctan function. If the place where the arctan is dicontinuous is chosen on the x-axes, then there is no problem: here is a numerical computation showing that the answer is $-50.2... = -16\pi$: There is a problem if the discontinuity is taken along the y-axes:

3 A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 4\cos^3(t), 4\sin^3(t) \rangle, 0 \leq t \leq 2\pi \ .$$

We can not do that directly. Guess which theorem to use!

Solution:
Take a vector field $\vec{F}(x,y) = \langle 0, x \rangle$ which has the curl 1. Then by Green the area is the line integral

$$\begin{aligned} & \int_0^{2\pi} \langle 0, 4\cos^3(t) \rangle \cdot \langle -12\cos^2(t)\ \sin(t), 12\sin^2(t)\cos(t) \rangle \ dt \\ = \ & 24 \int_0^{2\pi} \left(\cos^2(t)\sin^2(t)\right)\left(\cos^2(t)\right) \ dt \\ = \ & 24 \int_0^{2\pi} \frac{\sin^2(2t)}{4} \left(\frac{\cos(2t)+1}{2}\right) \\ \Rightarrow \ & 6\pi . \end{aligned}$$

4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise. (You might

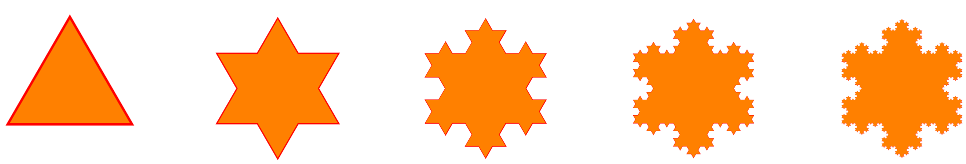
have seen in class the computation of the area of an ellipse using Green, you can use that).

Solution:
P and Q have continuous partial derivatives on $\mathbf{R^2}$, so by Green's Theorem we have 2 times the area 40π of the ellipse. But because the curve is oriented clockwise, the result is -80π.

5 Use Green's Theorem to evaluate

$$\int_C \langle \sin(\sqrt{1+x^3}), 7x \rangle \, d\vec{r} \,,$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length $1/3$ added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length $1/9$ added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length $1/27$ added and $K(4)$ is $K(3)$ with $3 * 4^3$ triangles of length $1/81$ added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26...$ which is between 1 and 2.



Solution:
Since $\text{curl}(F) = 7$, we have to compute the area of $K(3)$ and multiply by 7. We have $|K(3)| = \left(\frac{\sqrt{3}}{4}\right)\left(1+\frac{3}{9}+\frac{12}{81}+\frac{48}{729}\right) = \frac{658\sqrt{3}}{243}$. This is the accepted answer!
If students dare: In the limit we have a geometric series $|K| = \frac{\sqrt{3}}{4}\left(1+\frac{3}{9}\sum_{k=0}^{\infty}\frac{4^k}{9^k}\right) = \frac{\sqrt{3}}{4}\left(1+\frac{3}{9}\left(\frac{1}{1-\frac{4}{9}}\right)\right) = \frac{2\sqrt{3}}{5}$.

Main points

The **curl** of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \ .$$

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector field and G is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_G \text{curl}(F)\ dxdy \ .$$