Homework 26: Theorem of line integrals

This homework is due Monday, 11/16 rsp Tuesday 11/17.

1 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x, y) = \langle x^2 + 2xy^2 + y, y^2 + 2x^2y + x \rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.

Solution:
(a) $f(x, y) = x^3/3 + y^3/3 + x^2y^2 + xy$,
(b) The integral equals $f(2, 8) - f(-1, 2) = 441$

2 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x, y) = \langle 3y^2/(1 + x^2), 6y \arctan(x) \rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = \langle t^2, 2t \rangle$ with $0 \leq t \leq 1$.

Solution:
(a) $f(x, y) = 3y^2 \arctan x$,
(b) The integral is $f(1, 2) - f(0, 0) = 3\pi - 0 = 3\pi$.

3 a) Find the work done by the force field $\vec{F}$ in moving an object from $P = (0, 1)$ to $Q = (2, 0)$.

$$\vec{F}(x, y) = \langle e^{-y}, -xe^{-y} \rangle; \ .$$

b) You swim in a field $\vec{F} = \langle x^8, \sin(y) \rangle$ along a path $\vec{r}(t) = \langle t, t^2 + \sin(\sin(\pi t)) \rangle$ from $t = 0$ to $t = 2$. Find the energy you have spent.

Solution:
(a) The vector field F is conservative with $F = \nabla f$, $f(x, y) = xe^{-y}$, therefore the work done is $f(2,0) - f(0,1) = 2$.
(b) The vector field F is conservative with $F = \nabla f$, $f(x, y) = x^9/9 - \cos y$. The energy is $f(2, 4) - f(0, 0) = (512/9 - \cos 4\pi) - (0 - 1) = 521/9 - \cos(4)$.

4 a) Verify that if $\vec{F} = \langle P, Q, R \rangle$ is conservative, then

$$P_y = Q_x, P_z = R_x, Q_z = R_y .$$

b) Is $\langle x^5 y, xy^2, zx \rangle$ conservative? If yes, find f such that $\vec{F} = \nabla f$, if not, give a reason.

Solution:
(a) The definition of a conservative vector field is one that is independent of the choice of path. By sticking to paths that lie in planes parallel to the coordinate planes, we get the result. For example, working in planes parallel to the xy-plane tells us that $P_y = Q_x$.
(b) No, since $P_y = x^5 \neq y^2 = Q_x$.

5 a) Show that the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

with $F(x, y, z) = \langle y, x, xyz \rangle$ is not conservative by using problem 4).
b) Find two different curves from $(0, 0, 0)$ to $(1, 1, 0)$ for which the line integral is different.

Solution:

(a) For example, $P_z = 0 \neq yz = R_x$.

(b) Unfortunately, we cannot take both curves that lie entirely in the xy-plane since $P_y = 1 = Q_x$. Consider a vertical line segment $I_{x,y}$ which joins $(x, y, 0)$ to $(x, y, 1)$. Then the integral over this segment is $\int_0^1 xyz\,dz = xy/2$. This gives us a hint how to proceed. Take γ_1 to be any curve from $(0, 0, 0)$ to $(1, 1, 0)$ in the xy-plane. Set γ_2 be the union of $I_{0,0}$ (oriented upwards), a curve in the plane $z = 1$ above γ_1 and $I_{1,1}$ (oriented downwards). The difference between the integrals over γ_2 and γ_1 is $xy/2|_{1,1} - xy/2|_{0,0} = 1/2$ so the two integrals are not the same.

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions. It tells that the work done along a path is the potential energy difference.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a)) \ .$$

This theorem can be used to dramatically simplify the computation of a line integral. Just find the potential f and evaluate the difference of potential values.

Recall that a region R is called **simply connected** if every closed loop in R can be pulled together to a point within R.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:
Gradient field $\leftrightarrow$ Conservative $\leftrightarrow$ Closed loop property
In simply connected open regions, these three properties are all equivalent to being irrotational $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$.