

## Homework 25: Line integrals

This homework is due Friday, 11/13 resp Tuesday 11/17.

- 1 a) Evaluate the line integral  $\int_C \vec{F} \cdot d\vec{r}$  if  $\vec{F}(x, y, z) = \langle x + y, y - z, z^2 \rangle$  and  $\vec{r}(t) = \langle t^2, t^3, t^2 \rangle$  with  $0 \leq t \leq 2$ .
- b) Evaluate the line integral  $\int_C \vec{F} \cdot d\vec{r}$  if  $\vec{F}(x, y, z) = \langle 3z, 3y, -3x \rangle$  and  $\vec{r}(t) = \langle t, \sin(t), \cos(t) \rangle$ ,  $0 \leq t \leq \pi$ .

### Solution:

(a) The integral is

$$\int_0^2 \langle t^2 + t^3, t^3 - t^2, t^4 \rangle \cdot \langle 2t, 3t^2, 2t \rangle dt = \int_0^2 (2t^3 - t^4 + 5t^5) dt$$

This evaluates to  $824/15$ .

(b) The integral is

$$\begin{aligned} & \int_0^\pi \langle \cos t, \sin t, -t \rangle \cdot \langle 1, \cos t, -\sin t \rangle dt = \\ & = \int_0^\pi (3 \cos t + 3 \sin t \cos t + 3t \sin t) dt = 3\pi . \end{aligned}$$

- 2 An electric current  $I$  produces a magnetic field  $\vec{B}$  whose flow lines are circles circling the wire. Let  $C : \langle r \cos(t), r \sin(t), 0 \rangle$ . Ampère's law is  $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$ , where  $\mu_0$  is a constant called permeability. Show that the magnitude  $B(r) = |\vec{B}|$  of the magnetic field at a distance  $r$  from the center of the wire is  $B = \frac{\mu_0 I}{2\pi r}$ . Note that  $B$  is a scalar function and  $\vec{B}$  is a vector field. Use first the information provided to find the vector field  $\vec{B}$ .

**Solution:**

The vector field has the form  $\vec{B} = B(r)\langle -y/r, x/r \rangle$  where  $B$  depends only on the radius  $r$ . Above  $\langle -y/r, x/r \rangle$  is the unit vector tangent to a circle of radius  $r$ . Integrating over a concentric circle of radius  $r$ , we get:  $\int_{C(r)} \vec{B} \cdot d\vec{r} = 2\pi B(r)$ . Solving for  $B(r)$ , we find that  $B(r) = \mu_0 I / (2\pi r)$  as desired.

- 3 Determine from each of the following cases, whether or not  $\vec{F}$  is conservative or not. If it is, find a function  $f$  such that  $\vec{F} = \nabla f$ .
- a)  $\vec{F}(x, y) = \langle 7e^x \sin(y), 7e^x \cos(y) \rangle$
  - b)  $\vec{F}(x, y) = \langle 3x^2 - 2y^2, 4xy + 3 \rangle$
  - c)  $\vec{F}(x, y, z) = \langle x + y, y + x, z^5 - \sin(z) \rangle$
  - d)  $\vec{F}(x, y, z) = \langle x, z, y \rangle$

**Solution:**

Check  $Q_x - P_y$  and integrate if necessary. a) yes, b) no, c) yes, d) no.

- 4 Evaluate the line integral  $\int_C \langle 1 - ye^{-x}, e^{-x} \rangle \cdot d\vec{r}$ , where  $C$  is the path  $\vec{r}(t) = \langle t, 1 + t + \sin(\sin(t)) \rangle$  and  $t$  is from 0 to  $\pi$ . You probably will have difficulty. A future "you" (who has a time machine) tells you that you can compute the integral also in a different way: find a function  $f$  which is a potential to the vector field, then evaluate  $f(\vec{r}(\pi)) - f(\vec{r}(0))$ . You can use this without justification for now. We will learn about this "warp" feature in the next lecture.

**Solution:**

Check first that the integrand is conservative:

$$\frac{\partial(1 - ye^{-x})}{\partial y} = -e^{-x} = \frac{\partial(e^{-x})}{\partial x}.$$

The function  $f = x + ye^{-x}$  satisfies  $\nabla f = \langle 1 - ye^{-x}, e^{-x} \rangle$ , hence the answer should be  $f(1, 2) - f(0, 1) = (1 + 2e^{-1}) - 1 = 2e^{-1}$ .

- 5 The topological notions appearing next are cool but not very essential for the course. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
- a)  $\{(x, y) \mid 0 < y < 3\}$ , b)  $\{(x, y) \mid 1 < |x| < 2\}$   
c)  $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$  d)  $\{(x, y) \mid (x, y) \neq (1, 2)\}$   
e)  $\{(x, y, z) \mid (x, y, z) \neq (1, 2, 3)\}$

**Solution:**

- a) The set is a horizontal strip which is open, connected and simply-connected.
- b) The set is open but is not connected because it has two components, one where  $x$  is positive, and one where  $x$  is negative.
- c) The set is a semi-annulus. It is connected and simply-connected; but since we have included the boundary, it is not open.
- d) The set is a plane minus a point. It is open and connected; but since it has a hole at  $(1, 2)$ , it is not simply-connected.

**Main definitions**

If  $\vec{F}$  is a vector field and  $C : t \mapsto \vec{r}(t)$  is a curve defined on the interval  $[a, b]$  then  $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$  is called the **line integral** of  $\vec{F}$  along the curve  $C$ .

A vector field is **conservative** in a region  $R$  if the line integral from  $A$  to  $B$  is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. It is **irrotational** if  $\text{curl}(F) = Q_x - P_y$  is zero everywhere in  $R$ .

A subset  $G$  of the plane is **open** if every point  $(x, y)$  in  $G$  is contained in a small disc centered at  $(x, y)$  which is also in  $G$ . A subset is **connected**, if one can connect any two points in  $G$  with a curve inside  $G$ . A subset  $G$  is **simply connected** if it is connected and every closed curve in  $G$  can be pulled together to a point within  $G$ .

**Clairaut test:** Zero curl is necessary for a gradient field.