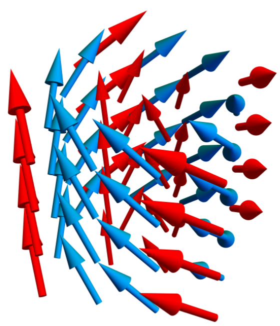
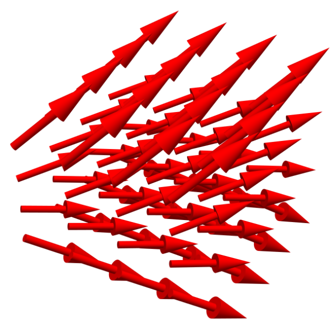
Homework 24: Vector fields

This homework is due Wednesday, 11/11 rsp Thursday 11/12.

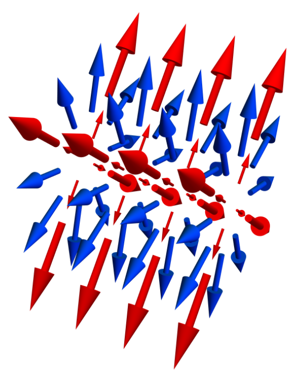
1 Match the vector fields $\vec{F}$ with the plots labeled A-D.
a) $\vec{F}(x, y, z) = \langle 0, y, z \rangle$, b) $\vec{F}(x, y, z) = \langle y, -x, 1 \rangle$, c) $\vec{F}(x, y, z) = \langle 1, 1, z \rangle$
d) $\vec{F}(x, y, z) = \langle x, y, z \rangle$



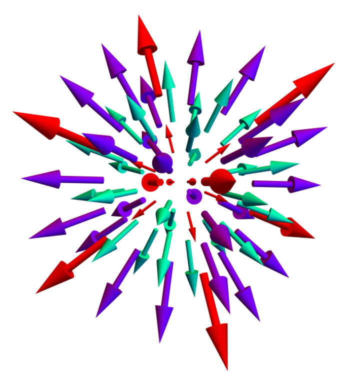
A



B



C



D

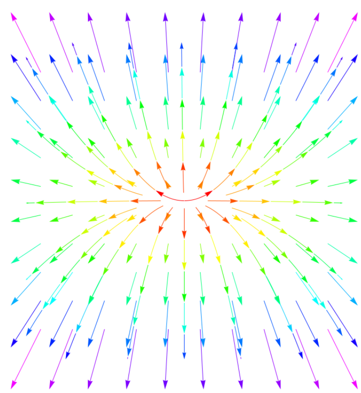
Solution:

2 a) Compute gradient vector field of f, where $f(x, y) = \log(7x - 5y)$ and log is the natural log.

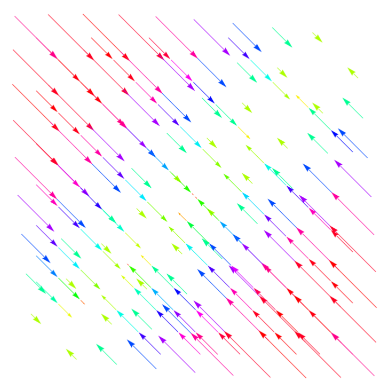
b) Given the vector field $\vec{F} = \langle P, Q \rangle = \langle \frac{x}{\sqrt{x^2-3y^2}} + 3y + 1, 3x - \frac{3y}{\sqrt{x^2-3y^2}} \rangle$. Check that $Q_x - P_y = 0$ and find a function $f(x, y)$ for which $\nabla f = \vec{F}$.

Solution:
(a) $\nabla f = \langle 7\sec^2(7x - 5y), -5\sec^2(7x - 5y) \rangle$. b) $f(x, y) = \sqrt{x^2 - 3y^2} + 3xy + x$.

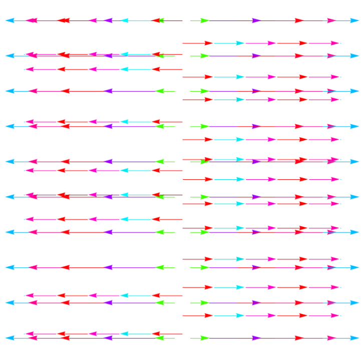
3 Match the functions f with the plots of their gradient fields labeled $A - D$. Give reasons for your choices.
a) $f(x, y) = \cos(x - y)$, b) $f(x, y) = \sin(\sqrt{1 + x^2})$
c) $f(x, y) = x^2 + 2y^2$, d) $f(x, y) = x(x + y)$



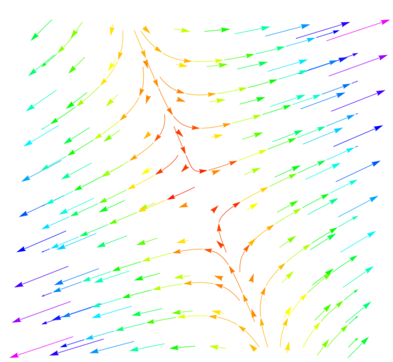
A



B



C



D

Solution:

For (c) and (d), the level sets of f are concentric circles. However, for (a), $|\nabla f|$ gets larger as $r = \sqrt{x^2 + y^2}$ increases, while in case of (b), the gradient oscillates in r. Thus, (c) is III and (a) is I.

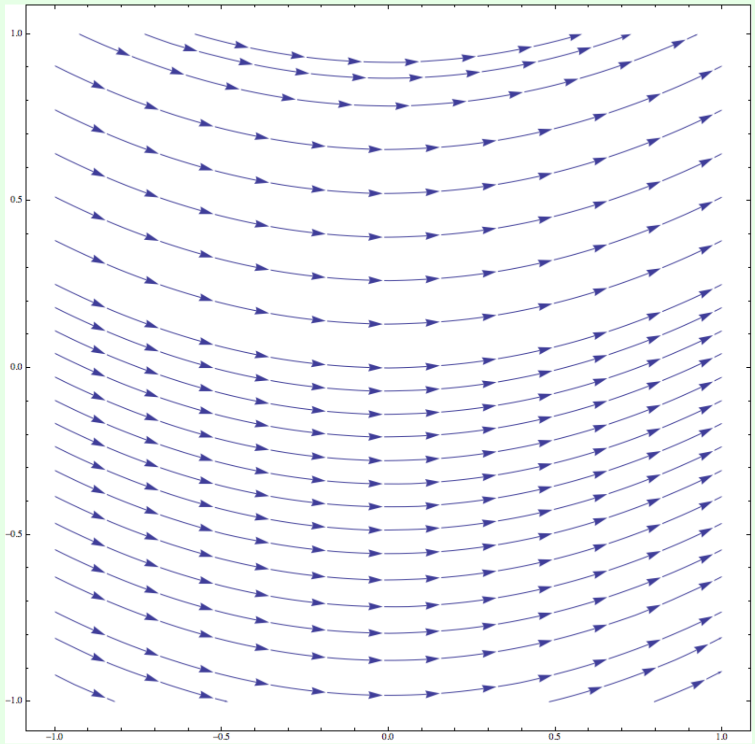
(b) $\nabla f = \langle 2x + y, x \rangle$, so IV.

(c) $\nabla f = \langle 2(x + y), 2(x + y) \rangle$ is constant when $x + y$ is constant, so II.

4 a) Sketch the vector field $\vec{F}(x, y) = \langle 2, x \rangle$ and then sketch some flow lines. What shape to these flow lines appear to have?

b) Find the flow line $\vec{r}(t)$ with $\vec{r}(0) = \langle 0, 0 \rangle$.

Solution:
The flow lines are parabolas $y = x^2/4 + C$. To see, integrate the gradient to find that: $x = 2t, y = t^2 + C$. In particular, the curve $\vec{r}(t)$ passing through $(0,0)$ is given by $(2t, t^2)$.



5 a) Let $\vec{F}(x,y) = \langle x, y\rangle(r^2 - 2r)$ with $r = |\langle x, y\rangle|$. Plot the vector field using Mathematica.
b) Make a stream plot of the field $\vec{F}(x,y) = \langle x^2 y, x + y\rangle$ using Mathematica. Example code is below.

```
VectorPlot[{x+y,x^2},{x,1,2},{y,-1,1}]
StreamPlot[{x+y,x^2},{x,1,2},{y,-1,1}]
VectorPlot3D[{x,y,z},{x,1,2},{y,1,2},{z,1,2}]
```

Solution: