

Homework 23: Cylindrical/Spherical integration

This homework is due Monday, 11/9 rsp Tuesday 11/10.

1 Evaluate the following integrals.

a) $\int_0^\pi \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta$

b) $\int_0^{2\pi} \int_{\pi/2}^\pi \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

Solution:

- a) The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \leq \theta \leq \pi, 0 \leq r \leq 2, 0 \leq z \leq 9 - r^2\}$. This represents the solid region in the first and second octants enclosed by the circular cylinder $r = 2$, bounded above by $z = 9 - r^2$, a circular paraboloid, and bounded below by the xy -plane. To evaluate:

$$\begin{aligned}\int_0^\pi \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta &= \int_0^\pi \int_0^2 [rz]_{z=0}^{z=9-r^2} \, dr \, d\theta \\&= \int_0^\pi \int_0^2 r(9 - r^2) \, dr \, d\theta \\&= \int_0^\pi d\theta \int_0^2 (9r - r^3) \, dr \\&= [\theta]_0^\pi \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^2 \\&= \pi(18 - 4) = 14\pi\end{aligned}$$

- b) The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \pi/2 \leq \phi \leq \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy -plane.

$$\begin{aligned}\int_0^{2\pi} \int_{\pi/2}^\pi \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} d\theta \int_{\pi/2}^\pi \sin \phi \, d\phi \int_1^2 \rho^2 \, d\rho \\&= [\theta]_0^{2\pi} [-\cos \phi]_{\pi/2}^\pi \left[\frac{1}{3}\rho^3 \right]_1^2 \\&= 2\pi (1) \left(\frac{7}{3} \right) = \frac{14\pi}{3}\end{aligned}$$

- 2 Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Solution:

In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, -\sqrt{4 - r^2} \leq z \leq \sqrt{4 - r^2}\}$. Thus the volume is

$$\begin{aligned}\iiint_E dV &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r\sqrt{4-r^2} \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2} \, dr \\ &= 2\pi \left[-\frac{2}{3}(4-r^2)^{3/2} \right]_0^1 \\ &= \frac{4}{3}\pi(8-3^{3/2})\end{aligned}$$

3 Use spherical coordinates to evaluate

$$\int \int \int_H (16 - x^2 - y^2) \, dV ,$$

where H is the solid hemisphere $x^2 + y^2 + z^2 \leq 16, z \geq 0$.

Solution:

In spherical coordinates, $H = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 4, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2\}$. Thus

$$\begin{aligned}
 & \iiint_H (16 - x^2 - y^2) \, dV \\
 &= \int_0^{\pi/2} \int_0^{2\pi} \int_0^4 [16 - (\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta)] \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \int_0^{\pi/2} \int_0^{2\pi} \int_0^4 (16 - \rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \int_0^{\pi/2} \int_0^{2\pi} \left[\frac{16\rho^3}{3} - \frac{\rho^5}{5} \sin^2 \phi \right]_{\rho=0}^{\rho=4} \sin \phi \, d\theta \, d\phi \\
 &= \int_0^{\pi/2} \int_0^{2\pi} \left(\frac{1024}{3} \sin \phi - \frac{1024}{5} \sin^3 \phi \right) \, d\theta \, d\phi \\
 &= \int_0^{2\pi} d\theta \int_0^{\pi/2} \left[-\frac{1024}{3} \cos \phi - \frac{1024}{5} (1 - \cos^2 \phi) \sin \phi \right] \\
 &= 2\pi \left[-\frac{1024}{3} \cos \phi - \frac{1024}{5} \left(\frac{\cos^3 \phi}{3} - \cos \phi \right) \right]_0^{\pi/2} \\
 &= 2\pi \left[0 + \frac{1024}{3} + \frac{1024}{5} \left(-\frac{2}{3} \right) \right] \\
 &= \frac{2048}{5} \pi
 \end{aligned}$$

- 4 Evaluate the integral by changing to spherical coordinates.

$$\int_{-a}^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2 z + y^2 z + z^3) \, dz \, dx \, dy$$

Solution:

The region of integration is the solid sphere $x^2 + y^2 + z^2 \leq a^2$, so $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq \rho \leq a$. Also $x^2z + y^2z + z^3 = (x^2 + y^2 + z^2)z = \rho^2z = \rho^3 \cos \phi$, so the integral becomes

$$\begin{aligned} & \int_0^\pi \int_0^{2\pi} \int_0^a (\rho^3 \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^\pi \sin \phi \cos \phi \, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^5 \, d\rho \\ &= \left[\frac{1}{2} \sin^2 \phi \right]_0^\pi [\theta]_0^{2\pi} \left[\frac{1}{6} \rho^6 \right]_0^a \\ &= 0 \end{aligned}$$

5 Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)} \, dx \, dy \, dz = 2\pi .$$

Solution:

The integral is

$$\lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^\pi \int_0^R \rho e^{-\rho^2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

We can separate out the θ and ϕ components. To integrate $\rho^3 e^{-\rho^2}$, we have to use integration by parts. Let $u = \rho^2$ and $dv = \rho e^{-\rho^2} d\rho$. We compute:

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \, d\phi \int_0^R \rho^3 e^{-\rho^2} d\rho \\ &= 2\pi(2) \left(\left[\rho^2 \left(-\frac{1}{2} \right) e^{-\rho^2} \right]_0^R - \int_0^R 2\rho \left(-\frac{1}{2} \right) e^{-\rho^2} d\rho \right) \\ &= \lim_{R \rightarrow \infty} 4\pi \left(-\frac{1}{2} R^2 e^{-R^2} + \left[-\frac{1}{2} e^{-\rho^2} \right]_0^R \right) \\ &= 4\pi \lim_{R \rightarrow \infty} \left[-\frac{1}{2} R^2 e^{-R^2} - \frac{1}{2} e^{-R^2} + \frac{1}{2} \right] \\ &= 4\pi \left(\frac{1}{2} \right) = 2\pi \end{aligned}$$

We have used that $R^2 e^{-R^2} \rightarrow 0$ as $R \rightarrow \infty$.

Main definitions

The integration factor in cylindrical coordinates $(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$ is r as in polar coordinates.

The integration factor in spherical coordinates $(x, y, z) = (\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))$ is $\rho^2 \sin(\phi)$. This was the surface area element $|\vec{r}_\phi \times \vec{r}_\theta|$

To evaluate an integral in spherical coordinates, we express the region in spherical coordinates, substitute $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$ and $z = \rho \cos(\phi)$ in the function and include the integration factor $\rho^2 \sin(\phi)$.