Homework 22: Triple integrals

This homework is due Friday, 11/6 rsp Tuesday 11/10.

1 a) Evaluate the iterated integral

$$\int_0^2 \int_0^z \int_0^{y^2} x \, dx \, dy \, dz \ .$$

b) Which of the two following scrambled versions make sense too?

$$\int_0^2 \int_0^{y^2} \int_0^z x \, dx \, dz \, dy \ .$$

$$\int_0^z \int_0^2 \int_0^{y^2} x \, dx \, dz \, dy \ .$$

Solution:
$64/60 = 16/15$

2 Evaluate the triple integral

$$\int \int \int_E 2yz \cos\left(x^5\right) dV \ ,$$

where

$$E = \{(x, y, z) \mid 0 \leq x \leq \pi/2, \ 0 \leq y \leq x, \ x \leq z \leq 2x\} \ .$$

Solution:
By the given description of E, we can write the triple integral as an iterated integral:

$$\iiint_E yz\cos(x^5)\,dV = \int_0^1\int_0^x\int_x^{2x} 2yz\cos(x^5)\,dz\,dy\,dx.$$

This we integrate in the usual way:

$$\begin{aligned}
\int_0^{\pi/2}\int_0^x\int_x^{2x} 2yz\cos(x^5)\,dz\,dy\,dx &= \int_0^{\pi/2}\int_0^x y\cos(x^5)\cdot\frac{1}{2}2z^2\Big|_x^{2x}\,dy\,dx\\
&= \int_0^{\pi/2}\int_0^x 2y\cos(x^5)\cdot\frac{1}{2}\cdot 3x^2\,dy\,dx\\
&= \frac{3}{2}\int_0^{\pi/2}\int_0^x x^2 2y\cos(x^5)\,dy\,dx\\
&= \frac{3}{2}\int_0^{\pi/2} x^2\cos(x^5)\cdot y^2\Big|_0^x\,dx\\
&= \frac{3}{2}\int_0^{\pi/2} x^2\cos(x^5)\cdot x^2\,dx\\
&= \frac{3}{4}\int_0^{\pi/2} 2x^4\cos(x^5)\,dx\\
&= \frac{6}{20}\sin(x^5)\Big|_0^{\pi/2}\\
&= \frac{6}{20}\sin(\pi^5/32)
\end{aligned}$$

The final answer is $\frac{3}{10}\sin(\pi^5/32)$.

3 Evaluate the triple integral

$$\int\int\int_E xy\,dV\ ,$$

where E is bounded by the parabolic cylinders $y = 3x^2$ and $x = 3y^2$ and the planes $z = 0$ and $z = x + y$.

Solution:

$$\iiint_E xy\, dV = \iint_R \int_0^{x+y} xy\, dz\, dA = \int_0^{1/3} \int_{3x^2}^{\sqrt{\frac{x}{3}}} \int_0^{x+y} xy\, dz\, dy\, dx$$

This is $\int_0^{1/3} \int_{3x^2}^{\sqrt{x}/3} xy(x+y)\ dz$. Evaluating the next integral gives

$$\int_0^{1/3} \frac{x^{5/2}}{9\sqrt{3}} - 9x^7 - \frac{9x^6}{2} + \frac{x^3}{6}\ dx = 1/2268 = 0.0004... .$$

4 Use a triple integral to find the volume of the given solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 25$.

Solution:
The paraboloid $x = y^2 + z^2$ intersects the plane $x = 25$ in the circle $y^2 + z^2 = 25$. Thus,

$$E = \{(x, y, z) \mid y^2 + z^2 \leq x \leq 25,\ y^2 + z^2 \leq 25\}$$

Let $D = \{(y, z) \mid\ y^2 + z^2 \leq 25\}$. Then using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we have

$$\begin{aligned}
\iint_D \left(\int_{y^2+z^2}^{25} dx\right) dA &= \iint_D \left(25 - (y^2+z^2)\right) dA \\
&= \int_0^{2\pi} \int_0^5 \left(25 - r^2\right) r\, dr\, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^5 \left(25r - r^3\right) dr \\
&= 2\pi \left[\frac{25r^2}{2} - \frac{r^4}{4}\right]_0^5 \\
&= \frac{625\pi}{2}
\end{aligned}$$

5 Find the moment of inertia

$$\int\int\int_E (x^2 + y^2)\ dxdydz$$

about the z-axis of the solid cone $E : \sqrt{x^2 + y^2} \leq z \leq 10$.

Solution:

$$\begin{aligned}
I_z &= \iiint_E \left(x^2 + y^2\right) \rho(x, y, z)\, dV \\
&= \iint_{x^2+y^2 \leq 10^2} \left[\int_{\sqrt{x^2+y^2}}^{10} \left(x^2 + y^2\right)\ dz\right] dA \\
&= \iint_{x^2+y^2 \leq 10^2} \left(x^2 + y^2\right) \left(10 - \sqrt{x^2 + y^2}\right) dA \\
&= \int_0^{2\pi} \int_0^{10} r^2\, (10 - r)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{10} \left(10r^3 - r^4\right)\ dr \\
&= (2\pi) \left[\frac{10}{4} r^4 - \frac{1}{5} r^5\right]_0^{10} \\
&= 10000\pi
\end{aligned}$$

Main definitions

If $f(x,y,z)$ is a function and E is a **solid**, then $\int\int\int_E f(x,y,z)\ dV$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(\frac{i}{n},\frac{j}{n},\frac{k}{n}) \in E} f(\frac{i}{n},\frac{j}{n},\frac{k}{n}) .$$

Like $dA = dxdy$ was a symbol for a small area, $dV = dxdydz$ indicates a small volume. Triple integrals are solved as a nested list of single integrals.

If $f(x,y,z) = 1$ then $\int\int\int_E 1\ dxdydz$ is the volume of the solid

A common situation is where the triple integral is reduced to a double integral

$$\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz]\ dxdy .$$

This is by far the most common case. For example, if $g(x,y) = 0$ and $f(x,y,z) = 1$, then

$$\int\int_R [\int_0^{h(x,y)} 1\ dz]\ dxdy = \int\int_R h(x,y)\ dxdy$$

is the signed volume of the solid under the graph of h. In variable calculus, where you were sometimes able to compute triple integrals by reducing to a single integral. You would have written $\int_a^b A(z)\ dz$ to compute the volume of a solid sandwiched between $z = a$ and $z = b$ for which the area of the cross section at height z is $A(z)$. In multi variable calculus we are much more flexible as we can now also reduce to a double integral.