Homework 21: Surface area

This homework is due Monday, 11/2 rsp Tuesday 11/3.

1 Find the surface area of the surface given by

$$z = \frac{2}{3}(x^{3/2} + y^{3/2}),\ 0 \leq x \leq 1,\ 0 \leq y \leq 1 .$$

Solution:
We compute: $f_x = x^{1/2}$ and $f_y = y^{1/2}$. The surface area is

$$\int_0^1 \int_0^1 \sqrt{1+x+y} \cdot dxdy = \frac{2}{3}\int_0^1 \{(2+y)^{3/2} - (1+y)^{3/2}\}dy$$

This evaluates to $-4/15(-1+8\sqrt{2}-9\sqrt{3})$.

2 Find the area of the surface given by the **helicoid**

$$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle \ .$$

with $0 \leq u \leq 1,\ 0 \leq v \leq \pi$.

Solution:
We compute:

$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle \qquad \vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle.$$

Therefore, $\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u\rangle$. The area is given by

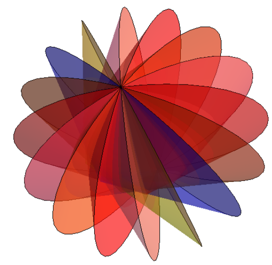
$$\int_0^\pi \int_0^1 \sqrt{1+u^2}dudv = \frac{\pi}{2}(\sqrt{2} + \log(1+\sqrt{2}))$$

To evaluate the integral

3 A decorative paper lantern is made of 8 surfaces. Each is parametrized by

$$\vec{r}(t,z) = \langle 10z\cos(t), 10z\sin(t), z\rangle$$

with $0 \le t \le 2\pi$ and $0 \le z \le 1$ and then translated or rotated. Find the total surface area of the lantern.



Solution:
We compute

$$\begin{aligned} \vec{r}_t &= \langle -10z\sin(t), 10z\cos(t), 0\rangle \\ \vec{r}_z &= \langle 10\cos(t), 10\sin(t), 1\rangle \\ \vec{r}_t \times \vec{r}_z &= \langle 10z\cos(t), 10z\sin(t), -100z\rangle \end{aligned}$$

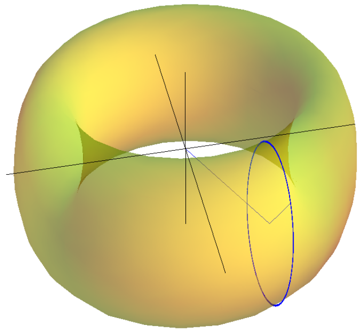
and the length is $|\vec{r}_t \times \vec{r}_z| = 10z\sqrt{101}$.

$$\int_0^{2\pi}\int_0^1 10z\sqrt{101}\, dzdt = 10\pi\sqrt{101}\ .$$

There are 8 pieces so that the final result is $\boxed{80\pi\sqrt{101}}$.

4 The figure shows the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b,0,0)$ and radius $a < b$. Parametric equations for the torus are

$$\begin{aligned} x &= b\cos\theta + a\cos\alpha\cos\theta \\ y &= b\sin\theta + a\cos\alpha\sin\theta \\ z &= a\sin\alpha\ , \end{aligned}$$

where θ and α are the angles shown in the figure. Find the surface area of the torus. 

Solution:
Let $\vec{r} = \langle x, y, z\rangle$. Then,

$$\vec{r}_\alpha = \langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha\rangle$$

and

$$\vec{r}_\theta = \langle -b\sin\theta - a\cos\alpha\sin\theta, b\cos\theta + a\cos\alpha\cos\theta, 0\rangle$$

Thus

$$\vec{r}_\alpha \times \vec{r}_\theta = \langle -ab\cos\alpha\cos\theta - a^2\cos^2\alpha\cos\theta,$$

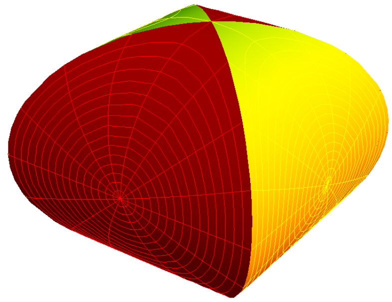
$$-ab\cos\alpha\sin\theta - a^2\cos^2\alpha\sin\theta, -ab\sin\alpha - a^2\cos\alpha\sin\alpha\rangle.$$

Therefore $|\vec{r}_\alpha \times \vec{r}_\theta| = a(b + a\cos\alpha)$ and so the surface area is

$$\int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)d\alpha d\theta = 2\pi \cdot 2ab\pi = 4\pi^2 ab.$$

5 The volume and surface area of the solid obtained by intersecting the solid cylinder $y^2+z^2 \leq 1$ with the solid cylinder $x^2+z^2 \leq 1$ has been found by Archimedes. Find the surface area of the surface

S bounding this solid.



Solution:

In the picture above, the boundary of the solid is made up of two red and two yellow pieces. The yellow piece facing us lies on the cylinder $x^2 + z^2$ between $|x| \geq |y|$. Thus, the surface area of *one* piece is

$$\int_{-1}^{1} \int_{-\arccos|y|}^{\arccos|y|} d\theta dy = \int_{-1}^{1} 2\arccos|y| dy = 4.$$

Thus the surface area is $4 \cdot 4 = 16$.

Main definitions:

A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \; .$$

Examples:

$\vec{r}(u,v)$	$\vert\vec{r}_u \times \vec{r}_v\vert$
$\langle \rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v) \rangle$	$\rho^2 \vert\sin(v)\vert$
$\langle u, v, f(u,v) \rangle$	$\sqrt{1 + f_u^2 + f_v^2}$
$\langle f(v)\cos(u), f(v)\sin(u), v \rangle$	$f(v)\sqrt{1 + f'(v)^2}$