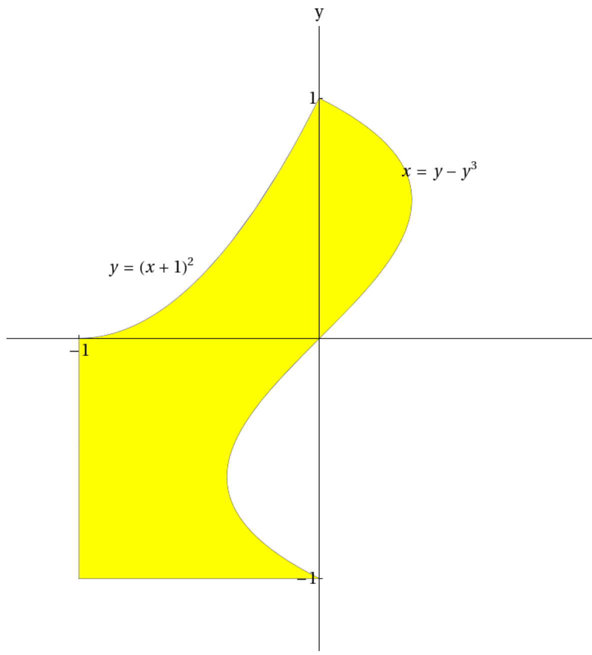
Homework 20: Polar integration

This homework is due Friday, 10/30 rsp Tuesday 11/3.

1 This is a review problem from the last section which does not deal with polar integration yet. Express the region R bound by the four curves $x = -1, y = -1, y = (x+1)^2, x = y - y^3$ as a union of type I or type II regions and evaluate the integral.

$$\iint_R y \, dA \; .$$



Solution:
Let us split the region in three parts:

$$\int_0^1 \int_0^{y-y^3} y dx dy = \int_0^1 y(y - y^3) dy = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

$$\int_{-1}^0 \int_0^{(x+1)^2} y dy dx = \int_{-1}^0 \frac{(x+1)^4}{2} dx = \frac{(x+1)^5}{10} \Big|_{-1}^0 = \frac{1}{10}.$$

$$\int_{-1}^0 \int_{-1}^{y-y^3} y dx dy = \int_{-1}^0 y(y - y^3 + 1) dy = -\frac{11}{30}.$$

The total is $-2/15$.

2 Evaluate the given integral by changing to polar coordinates:

$$\iint_R 6x \, dA \; ,$$

where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

Solution:
In polar coordinates, the first equation of the first circle is $r = 2$ while the second circle is $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$. Using the fact that $2\cos\theta < 2$, the integral over D is given by

$$6\int_0^{\pi/2} d\theta \int_{2\cos\theta}^{2} (r\cos\theta)rdr = 6\int_0^{\pi/2} \cos\theta d\theta \cdot \frac{8 - 8\cos^3\theta}{3} \ .$$

Using that the integral $\int_0^{\pi/2} \cos\theta d\theta = 1$ and $\int_0^{\pi/2} \cos^4\theta d\theta = 3\pi/16$, we see that

$$\iint_D 6x dA = 6\frac{8}{3} - \frac{8}{3} \cdot \frac{3\pi}{16} = 6(8/3 - \pi/2) = 16 - 3\pi \ .$$

3 Use polar coordinates to find the volume of the solid bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Solution:
In cylindrical coordinates, the equations of the paraboloids become $z = 3r^2$ and $z = 4 - r^2$. Since the first paraboloid is convex, while the second one is concave, the solid in question is above the first paraboloid and below the second. The two meet in the circle given by the equations $r = 1, z = 1$. Therefore, we must integrate

$$\int_0^{2\pi} d\theta \int_0^1 rdr \int_{3r^2}^{4-r^2} 1dz = 2\pi \int_0^1 r(4 - 4r^2) = 2\pi(4 - 3).$$

This simplifies to 2π.

4 Let D be the disk with center the origin and radius a. What is the average distance from points in D to the origin?

Solution:
We work in polar coordinates.

$$\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r\cdot r dr d\theta = \frac{1}{\pi a^2}\cdot 2\pi\cdot a^3/3 = 2a/3.$$

5 Evaluate the iterated integral

$$\int_0^2\int_0^{\sqrt{2x-x^2}} 9\sqrt{x^2+y^2}\ dy\, dx\ .$$

Solution:
The integrand is $9\sqrt{x^2+y^2} = 9r$. The region in question is a semicircle centered at 1 with radius 1. If $x = r\cos\theta$ and $y = r\sin\theta$, then $y = \sqrt{2x-x^2}$ simplifies to $r = 2\cos\theta$. Thus the integral is

$$\int_0^{\pi/2}\int_0^{2\cos\theta} 9r\cdot r dr d\theta = \int_0^{\pi/2} 72/3\cdot\cos^3\theta d\theta = 16.$$

Main definitions

Polar coordinates $(x, y) = (r\cos(t), r\sin(t))$ allow to describe regions bound by polar curves $(r(\theta), \theta)$.

The **average** of a quantity $f(x, y)$ over a region G is the fraction

$$\frac{\int\int_G f(x,y)\ dA}{\int\int_G 1\ dA} .$$

To integrate in polar coordinates, we evaluate the integral

$$\int\int_R f(x,y)\ dxdy = \int\int_R f(r\cos(\theta), r\sin(\theta))r\ drd\theta$$

where R is described in polar coordinates.