Homework 2: Vectors and Dot product

This homework is due Monday, 9/14 rsp Tuesday 9/15.

1 A kite is pulled with a force $\vec{F} = \langle 2, 1, 4 \rangle$. It has velocity $\vec{v} = \langle 1, -1, 1 \rangle$. The dot product of $\vec{F}$ with $\vec{v}$ is called power.
a) Find the angle between the force and the velocity.
b) Find the vector projection of the force onto the velocity vector.

Solution:
(a) To find the angle between the force and velocity, we make use of the formula:

$$\cos\theta = \frac{F \cdot v}{|F||v|}.$$

The magnitude of the force is $\sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$. The magnitude of the velocity is $\sqrt{3}$. The dot product $F \cdot v = 5$. Thus,

$$\cos\theta = \frac{F \cdot v}{|F||v|} = \frac{5}{\sqrt{21} \cdot \sqrt{3}}$$

Hence, $\theta = \arccos(\frac{5}{\sqrt{63}})$.
(b) The projection of $\vec{F}$ onto $\vec{v}$ is given by

$$proj_v F = \frac{F \cdot v}{|v|^2} v = \frac{5}{\sqrt{3}^2}\langle 1, -1, 1 \rangle = \frac{5}{3}\langle 1, -1, 1 \rangle.$$

2 Light shines long the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is $-\vec{a}$.
Hint. Reflect first at the xy-plane and find the components of

the ray after reflection. You can assume that in that case, the reflected vector is in the plane spanned by $\vec{k} = \langle 0, 0, 1 \rangle$ and $\vec{a}$.

Solution:
If we reflect at the xy-plane, then the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ gets changed to $\langle a_1, a_2, -a_3 \rangle$. You can see this by watching the reflection from above. Notice that the first two components stay the same. Do the same process with the other planes. If we next reflect $\langle a_1, a_2, -a_3 \rangle$ off the xz-plane, we get $\langle a_1, -a_2, -a_3 \rangle$. Finally, reflecting this vector off the yz-plane, we get the vector $\langle -a_1, -a_2, -a_3 \rangle$ as desired.

3 Given the vectors $\vec{a} = \langle 1, 2, 1 \rangle, \vec{b} = \langle 1, -1, 1 \rangle, \vec{c} = \langle 0, 1, 1 \rangle, \vec{d} = \langle 2, 3, 4 \rangle$, compute all possible dot products and determine which pairs are perpendicular.

Solution:
We compute:

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= (1)(1) + (2)(-1) + (1)(1) = 1 - 2 + 1 = 0 \\
\vec{a} \cdot \vec{c} &= (1)(0) + (2)(1) + (1)(1) = 0 + 2 + 1 = 3 \\
\vec{a} \cdot \vec{d} &= (1)(2) + (2)(3) + (1)(4) = 2 + 6 + 4 = 12 \\
\vec{b} \cdot \vec{c} &= (1)(0) + (-1)(1) + (1)(1) = 0 - 1 + 1 = 0 \\
\vec{b} \cdot \vec{d} &= (1)(2) + (-1)(3) + (1)(4) = 2 - 3 + 4 = 3 \\
\vec{c} \cdot \vec{d} &= (0)(2) + (1)(3) + (1)(4) = 3 + 4 = 7
\end{aligned}$$

The dot product of perpendicular vectors is 0. Thus, we conclude that $\vec{a} \perp \vec{b}$ and $\vec{b} \perp \vec{c}$.

4 a) Find the angle between a diagonal of a cube and the diagonal

in one of its faces.

b) The hypercube or tesseract has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, 1)$.

Solution:

(a) Consider the 'standard' cube with vertices whose coordinates are either 0 or 1. The line segment from $(0, 0, 0)$ to $(0, 1, 1)$ is a diagonal of one of its faces (the face on the yz-plane), while the line segment from $(0, 0, 0)$ to $(1, 1, 1)$ is a diagonal of the cube. These line segments can be expressed as the vectors $\langle 0, 1, 1\rangle$ and $\langle 1, 1, 1\rangle$, respectively. The cosine of the angle between them is then

$$\cos\theta = \frac{\langle 0, 1, 1\rangle \cdot \langle 1, 1, 1\rangle}{|\langle 0, 1, 1\rangle| \cdot |\langle 1, 1, 1\rangle|} = \frac{2}{\sqrt{2} \cdot \sqrt{3}} = \frac{2}{\sqrt{6}}.$$

Hence $\theta = \arccos(\frac{2}{\sqrt{6}})$.

(b) The vector connecting $(1, 1, 1, 1)$ to $(-1, -1, -1, -1)$ is $\langle 2, 2, 2, 2\rangle$. The vector connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, 1)$ is $\langle 2, 2, 2, 0\rangle$. Thus, the cosine of the angle between the hyper diagonal and the space diagonal is given by

$$\cos\theta = \frac{\langle 2, 2, 2, 2\rangle \cdot \langle 2, 2, 2, 0\rangle}{|\langle 2, 2, 2, 2\rangle||\langle 2, 2, 2, 0\rangle|} = \frac{12}{\sqrt{16} \cdot \sqrt{12}} = \frac{\sqrt{3}}{2}.$$

Finally, we calculate $\theta = \arccos(\frac{\sqrt{3}}{2})$.

5 a) Verify that if $\vec{a}, \vec{b}$ are nonzero, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}, \vec{b}$ if $\vec{c}$ is not zero.

b) Verify the parallelogram law $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2+2|\vec{b}|^2$.

Solution:

(a) The cosine of the angle between a and c is

$$\begin{aligned}\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|} &= \frac{\vec{a}\cdot(|\vec{a}|\vec{b}+|\vec{b}|\vec{a})}{|\vec{a}|\cdot|\vec{c}|} \\ &= \frac{|\vec{a}|(\vec{a}\cdot\vec{b})+|\vec{a}|^2|\vec{b}|)}{|\vec{a}|\cdot|\vec{c}|} = \frac{(\vec{a}\cdot\vec{b})+|\vec{a}||\vec{b}|}{|\vec{c}|}.\end{aligned}$$

A similar calculation yields that the cosine of the angle between b and c is $\frac{(\vec{b}\cdot\vec{a}+|\vec{b}||\vec{a}|}{|\vec{c}|}$. Because these two expressions are exactly the same, this tells us that either c bisects the angle between a and b or that a and b are colinear. However, if a and b are colinear, c is also colinear with a and b and thus it still bisects the angle between them.

(b) We compute:

$$\begin{aligned} & |\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})+(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \\ =\ & (\vec{a}\cdot a+2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b})+(\vec{a}\cdot\vec{a}-2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}) \\ =\ & 2(\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}) \\ =\ & 2(|\vec{a}|^2+|\vec{b}|^2) \\ =\ & 2|\vec{a}|^2+2|\vec{b}|^2.\end{aligned}$$

Main definitions

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a **vector** $\vec{v} = \langle x - a, y - b, z - c \rangle$. As it connects P with Q, we also write $\vec{v} = \vec{PQ}$. The real numbers v_1, v_2, v_3 in $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are called the **components** of $\vec{v}$. The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. The **addition** of two vectors is $\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$. The **scalar multiple** $\lambda \vec{u} = \lambda \langle u_1, u_2, u_3 \rangle = \langle \lambda u_1, \lambda u_2, \lambda u_3 \rangle$. The difference $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$. The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ satisfying $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$. The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$. **Pythagoras tells:** if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.