Homework 19: Double integrals

This homework is due Wednesday, 10/28 rsp Thursday 10/29.

1 a) (4 points) Find the iterated integral

$$\int_0^1 \int_0^2 6xy/\sqrt{x^2+2y^2}\, dy\, dx\ .$$

b) (4 points) Now compute

$$\int_0^1 \int_0^2 6xy/\sqrt{x^2+2y^2}\, dx\, dy\ .$$

c) (2 points) Wouldn't Fubini assure that a) and b) are the same? What change would be needed in b) to make the results agree?

Solution:

a) $\int_0^1 \int_0^2 6xy/\sqrt{x^2+y^2}\, dy\, dx = 26 - 16\sqrt{2}$.

b) $\int_0^2 \int_0^1 6xy/\sqrt{x^2+y^2}\, dy\, dx = -8 - \sqrt{2} + 3\sqrt{6}$.

c) One would also have to change $dydx$ to $dxdy$.

2 Calculate the double integral

$$\int\int_R \frac{6x}{1+xy}\, dA$$

over the region $R = \{(x,y) \mid 0 \le y \le x, 0 \le x \le 1\ \}$.

Solution:

Integrate using substitution and partial fractions $3(\log(4) + \pi - 4)$.

3 a) Evaluate the double integral

$$\int_0^2 \int_y^{2y} 3xy \, dx \, dy \ .$$

b) Evaluate the integral of xy over the same region but as a type I integral (which needs to be split into two integrals):

$$\int_0^2 \int_{\dots}^{\dots} 3xy \ dydx + \int_2^4 \int_{\dots}^{\dots} 3xy \ dydx \ .$$

Solution:

a)

$$\begin{aligned} \int_0^2 \int_y^{2y} 3xy \, dx \, dy &= \int_0^2 \frac{1}{2} 3x^2 y \Big|_{x=y}^{x=2y} \, dy \\ &= \frac{1}{2} \int_0^2 3y(4y^2 - y^2) \, dy = \frac{1}{2} \int_0^2 9y^3 \, dy \\ &= \frac{3}{2} (\frac{1}{4} 3y^4) \Big|_0^2 = \frac{3}{2} 3(4-0) = 18. \end{aligned}$$

b) Make a picture to get the region

$$\int_0^2 \int_{x/2}^{x} xy \ dydx + \int_2^4 \int_{x/2}^{2} xy \ dydx \ .$$

4 We want to find the volume of the region below the graph of $f(x,y) = 15xy^2$, where R is the region enclosed by the curves $x = 0, x = \sqrt{1-y^2}$. In other words, find $\int\int_R 15xy^2 \, dA$.

Solution:

$$\begin{aligned}
\int\int_R xy^2\,dA &= \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} 15xy^2\,dx\,dy \\
&= \int_{-1}^{1} y^2\,(\frac{15}{2}x^2)\Big|_{x=0}^{x=\sqrt{1-y^2}}\,dy = \frac{15}{2}\int_{-1}^{1} y^2(1-y^2)\,dy \\
&= \frac{15}{2}\int_{-1}^{1}(y^2-y^4)\,dy = \frac{15}{2}\,(\frac{1}{3}y^3-\frac{15}{5}y^5)\Big|_{-1}^{1} \\
&= \frac{15}{2}(\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{5}) = \frac{30}{15} = 2
\end{aligned}$$

5 Evaluate the integral:

$$\int_0^8\int_{\sqrt[3]{y}}^2 4e^{x^4}\,dx\,dy\ .$$

Solution:

$$\begin{aligned}
4\int_0^8\int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy &= 4\int_0^2\int_0^{x^3} e^{x^4}\,dy\,dx \\
&= 4\int_0^2 e^{x^4}[y]_{y=0}^{y=x^3}\,dx = 4\int_0^2 x^3e^{x^4}\,dx \\
&= e^{x^4}]_0^2 = (e^{16}-1)
\end{aligned}$$

Main definitions

If R is a two dimensional region and $f(x,y)$ a function of two variables, the **double integral** $\int\int_R f(x,y)\,dA$ is defined the limit of the Riemann sum $(1/n^2)\sum_{(i/n,j/n)\in R} f(i/n,j/n)$ for $n\to\infty$. Depending on how we order the sum we write $dA = dxdy$ or $dA = dydx$ and reduce to single variable integrals.

A **type I region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \} .$$

An integral over such a region is called a **type I integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx .$$

A **type II region** is of the form

$$R = \{(x,y) \mid c \le y \le d, a(y) \le x \le b(y)\ \} .$$

An integral over such a region is called a **type II integral**

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$

Fubini's theorem allows to switch the order of integration over a rectangle, if the function f is continuous:

$$\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx .$$