Homework 18: Global extrema

This homework is due Monday, 10/26 rsp Tuesday 10/27.

1 Find the extreme values of f on the region described by the inequality.

$$f(x,y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16 \ .$$

Solution:
We need to use Lagrange multipliers for the boundary and partial derivatives to determine critical points from the interior. Let's consider the interior first. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x-4, 6y \rangle = \langle 0,0 \rangle \Rightarrow x = 1, y = 0$. So $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. $f(x,y) = 2x^2 + 3y^2 - 4x - 5$, $g(x,y) = x^2 + y^2 = 16 \Rightarrow$

$$\begin{aligned} \nabla f &= \langle 4x-4, 6y \rangle, \\ \lambda \nabla g &= \langle 2\lambda x, 2\lambda y \rangle \\ 4x - 4 &= 2\lambda x, \\ 6y &= 2\lambda y \end{aligned}$$

So if $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f(-2, \pm 2\sqrt{3}) = 47$. So the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1,0) = -7$.

2 a) We suppose that the Cobb Douglas production formula $Q(L,K) = L^{1/4}K^{3/4} = 100$, which tells that the quantity Q is constant. What values of L and K minimizes the cost function $C(L,K) = 4L + 5K$?
b) Is there a global maximum or minimum on $L \geq 0, K \geq 0$ without the constraint $Q = 100$? If yes, what is the maximum, or what is the minimum?

Solution:
$C(L,K) = 4L + 5K$, $g(L,K) = L^{1/4}K^{3/4} = 100 \Rightarrow \nabla C = \langle 4,5 \rangle$, $\lambda \nabla g = \langle \lambda \frac{1}{4} b L^{-3/4}K^{3/4}, \lambda \frac{3}{4} L^{1/4}K^{-1/4} \rangle$. Then $\dfrac{4}{1/4}(\dfrac{L}{K})^{3/4} = \dfrac{5}{3/4}(\dfrac{K}{L})^{1/4}$ and $L^{1/4}K^{3/4} = 100 \Rightarrow \frac{5/4}{4(3/4)} = (\frac{L}{K})^{3/4}(\frac{L}{K})^{1/4} \Rightarrow L = \frac{K(5/4)}{4(3/4)}$. Plugging this into the production formula, we get $L = [\dfrac{K(5/4)}{4(3/4)}]^{1/4}K^{3/4} = 100$. Hence $K = \dfrac{100(4)^{1/4}(3/4)^{1/4}}{(5)^{1/4}(1/4)^{1/4}} = \dfrac{100}{1}(\dfrac{12}{5})^{1/4}$ and $L = \dfrac{100(5)^{3/4}(1/4)^{3/4}}{4^{3/4}(3/4)^{3/4}} = \dfrac{100}{1}(\dfrac{5}{12})^{3/4}$. There is a minimum 0 at $L = 0$ or $K = 0$. There is no global maximum. In the original problem, we had $L \geq 0, K \leq 0$ and there, there is **no maximum and no minimum**.

3 Find the absolute maximum and minimum values of

$$f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2);$$

on the disk $D = \{x^2 + y^2 \leq 4\}$.

Solution:
Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0, 0), (0, \pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0, 0) = 0, f(\pm 1, 0) = e^{-1}$ and $F(0, \pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x, y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4} = 0.146....$ Thus on D the global maximum of f is $f(0, \pm 1) = 2e^{-1} = 0.73..$ and the global minimum is $f(0, 0) = 0$.

4 a) Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint

$$f(x, y) = \frac{1}{x} + \frac{1}{y}; \; g(x, y) = \frac{1}{x^2} + \frac{1}{y^2} = 1 \, .$$

b) Is there a global maximum or global minimum? If not, why does this not violate the Bolzano theorem?

Solution:

a) $f(x,y) = \frac{1}{x} + \frac{1}{y}$, $g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $\frac{1}{x^2} + \frac{1}{y^2} = \frac{2}{x^2} = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

b) There is a global minimum and maximum. there is a global maximum on $(1/\sqrt{2}, 1/\sqrt{2})$ and a global minimum on $(-1/\sqrt{2}, -1/\sqrt{2})$. There are four branches to the constraint $1/x^2 + 1/y^2 = 1$. In the first quadrant the function f is maximal at $(1/\sqrt{2}, 1/\sqrt{2})$ and goes to 1 when either x goes to infinity or y goes to infinity. In the third quadrant, f is minimal at $(-1/\sqrt{2}, -1/\sqrt{2})$ and goes to -1 as x goes to infinity or minus infinity. on the second and third quadrant, the function is monotone going from -1 to 1.

5 A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108. Find the dimensions of the package with largest volume $V(x,y,z) = xyz$ that can be mailed under the constraint $x+2y+2z \leq 108$.

Solution:
We want to extremize $V = xyz$ where $x + 2y + 2z \leq 108$, $x \geq 0$, $y \geq 0$, $z \geq 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11,664$ cubic units. The maximal dimension is $(36, 18, 18)$.
At the boundary , where $x = 0$ or $y = 0$ or $z = 0$, we get 0 as a volume.

Main definitions:

Standard assumption is still that all functions have continuous first and second derivatives.

A point (x_0, y_0) is an **absolute maximum** = **global maximum** on a domain R, if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R.

To find a global maximum, we look at the local maxima and minima as well as the maxima and minima on the boundary. The later is a Lagrange problem. If the domain is unbounded, we also have to look at the behavior of the function when $x, y \to \infty$.

Example.
$f(x, y) = x^2 + y^2 - x^4 - y^4$ has a local minimum at $(0, 0)$ but this is not a global minimum because $f(1000, 1000)$ for example

is smaller than $f(0,0) = 0$.

If $f(x,y)$ is considered on the domain $R = \{x^2 + y^2 \leq 1\}$ then the situation has changed and we need to look at extrema on the boundary too. One can see that $(0,0), (\pm 1, 0), (0, \pm 1)$ are all global minima.

Bolzano theorem: A region R is bounded if there is r such that R is contained in a disc of radius r. A region is closed if it contains all boundary points. A continuous function on a region which is bounded and closed always has a global maximum and a global minimum on R.