Homework 17: Lagrange multipliers

This homework is due Friday, 10/23 rsp Tuesday 10/27.

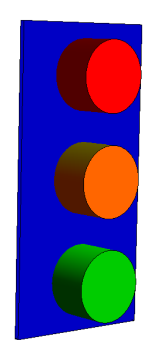
1 Use Lagrange multipliers to find the maximal value of the function $f(x, y) = 3e^{xy}$ subject to the constraint $x^3 + y^3 = 16$.

Solution:
By the method of Lagrange multipliers,

$$\langle 3ye^{xy}, 3xe^{yx} \rangle = \lambda \langle 3x^2, 3y^2 \rangle$$

which tells us that $y = Cx^2$ and $x = Cy^2$ where $C = \lambda/e^{xy}$. The two equations tell us that $y = C^3y^4$ which translates to $y = 0$ or $y^3 = C^3$. But if $y = 0$, then $x = 0$ which violates the constraint. Thus, $y^3 = C^3$. Similarly, $x^3 = C^3$. Hence $x = y$. Plugging into the constraint, we find $\boxed{(x, y) = (2, 2)}$. Hence $f(2, 2) = 3e^4$ is a **local maximum**. To see that it is a maximum, one could either look at the graph of the function $f(x, (16 - x^3)^{1/3})$ or find numerically values nearby.

2 The material to build a traffic light is $g(x, y) = 6 + 6\pi xy + 3\pi x^2 = 12$ is fixed (the radius of each cylinder is x and the height is y and the constant 6 is the material for the back plate). We want to build a light for which the shaded region with volume $f(x, y) = 3\pi x^2 y$ is maximal. Use the Lagrange method.



Solution:
The Lagrange equations $\nabla f = \lambda \nabla g, g = 12$ are

$$\begin{aligned} 6\pi xy &= \lambda(6\pi y + 6\pi x) \\ 3\pi x^2 &= \lambda(6\pi x) \\ 2\pi xy + \pi x^2 &= 6 \end{aligned}$$

Eliminating λ from the first two equations gives $x = y$. Plugging into the constraint gives $\boxed{x = \sqrt{2/(3\pi)} = y}$. The maximal value is $3\pi x^2 y = 2\sqrt{2/(3\pi)} = 0.921....$

3 Use Lagrange multipliers to find the maximum and minimum f under the two constraints:

$$\begin{aligned} f(x, y, z) &= 3x - y - 3z; \\ g(x, y, z) &= x + y - z = 0 \\ h(x, y, z) &= x^2 + 2z^2 = 1 \ . \end{aligned}$$

Solution:
By the method of Lagrange multipliers,

$$\langle 3, -1, -3 \rangle = \lambda \langle 1, 1, -1 \rangle + \mu \langle 2x, 0, 4z \rangle$$

The "middle" equation tells us that $\lambda = -1$. Hence, $2x\mu - 1 = 3$ and $4z\mu + 1 = -3$. In particular, $x\mu = 2$ and $-z\mu = 1$ so $\boxed{x = -2z}$. Plugging $x = -2z$ into the first constraint, we find $\boxed{y = 3z}$. Plugging $x = -2z$ in the second constraint, we see that $6z^2 = 1$. Thus the two critical points are $(\frac{2}{\sqrt{3}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$ and $(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The first gives the maximal value of $\frac{12}{\sqrt{6}}$ while the second gives the minimum value of $-\frac{12}{\sqrt{6}}$.

4 Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral. *Hint:* Use Heron's formula for the area:

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

Solution:
To maximize the area A, it is sufficient to maximize A^2. We do so since the formula for A^2 does not involve a square root. If the perimeter p is fixed, then so is s. Hence, it is sufficient to maximize

$$\begin{aligned} f(x, y, z) &= s(s-x)(s-y)(s-z) \\ &= (y+z-x)(z+x-y)(x+y-z) \end{aligned}$$

provided $g(x, y, z) = x + y + z = 2s$. By Lagrange multipliers,

$$\nabla f = \lambda\langle 1, 1, 1\rangle.$$

Thus, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$. We compute:

$$0 = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = (x+y-z) \cdot -4(x-y)$$

By the triangle inequality, $x+y > z$, so $x = y$. Similarly, $x = z$ forcing the triangle to be equilateral.

5 Which pyramid of height h over a square $[-a, a]\times[-a, a]$ with surface area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$ has maximal volume $V(h, a) = 4ha^2/3$? By using new variables (x, y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$. Use the later variables.

Solution:
An elegant solution can be obtained by first simplifying the constraint and write $x^2(y^2 - x^2) - (1 - x^2)^2 = x^2y^2 + 2x^2 - 1 = 0$. Now the Lagrange equations are not so bad:

$$\begin{aligned} 2xy &= \lambda(4x + 2xy^2) \\ x^2 &= \lambda 2x^2 y \\ x^2y^2 + 2x^2 &= 1 \end{aligned}$$

Eliminating λ by cross multiplying the first two equations gives $4x^3y^2 = x^2(4x + 2xy^2)$. Since $x = 0$ is not possible, we can divide both sides by x^3 and get $y^2 = 2$. Plugging into the constraint, we get $x = 1/2$. Without this simplification, the Lagrange system would have become more complicated:

$$\begin{aligned} 2xy &= \lambda(\sqrt{y^2 + x^2} + x^2/\sqrt{y^2 + x^2} + 2x) \\ x^2 &= \lambda yx/\sqrt{y^2 + x^2} \\ 1 &= x\sqrt{y^2 + x^2} + x^2 \end{aligned}$$

Main definitions

The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ for the three unknowns x, y, λ are called **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0$ are the **Lagrange equations** in three dimensions.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), g(x,y,z) = 0, h(x,y,z) = 0$ are the **Lagrange equations** in three dimensions with two constraints. There are two Lagrange multipliers λ, μ.

Lagrange theorem: Maxima or minima of f on the constraint $g = c$ are either solutions of the Lagrange equations or critical points of g.