Homework 15: Directional Derivatives

This homework is due Monday, 10/19 rsp Tuesday 10/20.

1 Find the gradient of

$$f(x,y,z) = \sqrt{x+yz}\ .$$

at the point $P = (1,3,1)$ and use it to find the rate of change of f at P in the direction of the vector $\vec{u} = \langle 2/7, 3/7, 6/7 \rangle$.

Solution:

(a) Gradient: $\nabla f(x,y,z) = \langle \frac{1}{2}(x+yz)^{-1/2}(1),\ \frac{1}{2}(x+yz)^{-1/2}(z),\ \frac{1}{2}(x+yz)^{-1/2}(y) \rangle = \langle \frac{1}{2\sqrt{x+yz}},\ \frac{z}{2\sqrt{x+yz}},\ \frac{y}{2\sqrt{x+yz}} \rangle$.

(b) $\nabla f(1,3,1) = \langle \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4} \rangle$.

(c) $D_u f(1,3,1) = \nabla f(1,3,1) \cdot \vec{u} = \langle \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4} \rangle \cdot \langle \frac{2}{7},\ \frac{3}{7},\ \frac{6}{7} \rangle = \frac{23}{28}$.

2 a) (5 points) Find the directional derivative of the function $f(x,y) = \log(x^2+y^2)$ at the point $P = (2,1)$ in the direction of the vector $\vec{v} = \langle -1, 2 \rangle$. (We use the notation $\log = \ln$).

b) (5 points) Find the directional derivative of $f(x,y,z) = xy + yz + zx$ at the point $P = (1,-1,3)$ in the direction from P to $Q = (2,4,5)$.

Solution:

(a) $\nabla f = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle$. At point $(2,1)$, it is $\langle 4/5, 2/5 \rangle$. The directional derivative is

$$\langle 4/5, 2/5 \rangle \cdot \frac{\langle -1, 2 \rangle}{|\langle -1, 2 \rangle|} = \frac{\langle 4/5, 2/5 \rangle \cdot \langle -1, 2 \rangle}{|\langle -1, 2 \rangle|} = 0.$$

(b) $\nabla f(x,y,z) = \langle y+z,\ x+z,\ y+x \rangle$, so $\nabla f(1,-1,3) = \langle 2,4,0 \rangle$. The unit vector in the direction of $\vec{PQ} = \langle 1,5,2 \rangle$ is $\vec{u} = \frac{1}{\sqrt{30}}\langle 1,5,2 \rangle$, so $D_u f(1,-1,3) = \nabla f(1,-1,3) \cdot \vec{u} = \langle 2,4,0 \rangle \cdot \frac{1}{\sqrt{30}}\langle 1,5,2 \rangle = \frac{22}{\sqrt{30}}$.

3 a) Find the direction of steepest descent for $f(x,y,z) = \frac{(x+y)}{z}$ at the point $P = (1,1,-1)$.
b) Find the value of the maximal rate of change at $(1,1,-1)$ in that direction found in a). This is the directional derivative in that direction.

Solution:

(a) The maximum rate of change occurs in the direction of the gradient. The gradient is

$$\nabla f = \langle \frac{1}{z}, \frac{1}{z}, -\frac{x+y}{z^2} \rangle, \qquad \nabla f(1,1,-1) = \langle -1,-1,-2 \rangle.$$

(b) The unit vector in the direction of the gradient is $\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \rangle$ and the rate of change is

$$\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \rangle \cdot \nabla f = \sqrt{6}.$$

4 Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.

Solution:
$f_x(x, y) = -y^2 e^{-xy}$, $f_y(x, y) = (1 - xy)e^{-xy}$ and $f_x(0, 2) = -4$, $f_y(0, 2) = 1$. if $\vec{u}$ is a unit vector which makes an angle θ with the positive x-axis, then $D_u f(0, 2) = f_x(0, 2) \cos\theta + f_y(0, 2) \sin\theta = -4 \cos\theta + \sin\theta$. We want $D_u f(0, 2) = 1$, so $-4 \cos\theta + \sin\theta = 1 \Rightarrow \sin\theta = 1 + 4 \cos\theta$. Square both sides, and use a trig identity: $1 - \cos^2\theta = 1 + 8 \cos\theta + 16 \cos^2\theta \Rightarrow 17 \cos^2\theta + 8 \cos\theta = 0 \Rightarrow \cos\theta = 0$ or $\cos\theta = -\frac{8}{17}$.
Now if $\cos\theta = 0$ then $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ but $\frac{3\pi}{2}$ does not satisfy the original equation. If $\cos\theta = -\frac{8}{17}$ then $\theta = \cos^{-1}(-\frac{8}{17})$ or $2\pi - \cos^{-1}(-\frac{8}{17})$ but $\cos^{-1}(-\frac{8}{17})$ is not a solution of the original equation. Thus the possible directions are either $\theta = \frac{\pi}{2}$ or $\theta = 2\pi - \cos^{-1}(-\frac{8}{17}) \approx 4.22$ rad.

5 [Arlington-Belmont-Waltham-Cambridge]
On http://goo.gl/fhY1rl,
you find a map of some suburbs of Boston (an original copy in office 432).
a) The map contains some creeks. Find an example which confirms the rule that water crosses level curves perpendicularly.
b) The map shows also some railway tracks. Identify a railway which follows the level curves of the height.
c) Find a point on the map where the slope is maximal in some direction. Estimate the steepness by counting level curves.

Solution:

a) Almost all creeks have this property. Water flows downwards. b) Railway tracks, for example from Cambridge to Belmont, going west have this property. c) Mt Pisrah in Winchester is quite steep. It does maybe 100 feet in 100 yards which is 30 yards per 100 yards. The slope is about 1/3. (This is a very rough estimate).

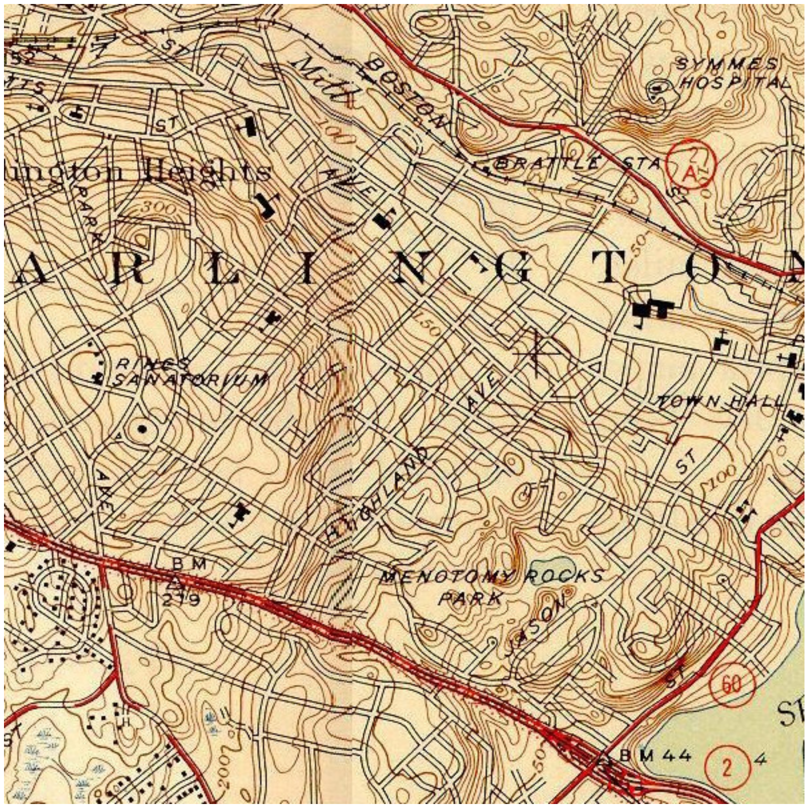
Main definition:

If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is the **directional derivative** of f in the direction $\vec{v}$.

For $\vec{v} = \nabla f / |\nabla f|$, the directional derivative is

$$D_v f = \nabla f \cdot \nabla f / |\nabla f| = |\nabla f| \ ,$$

so that f **increases** in the direction of the gradient. The value $|\nabla f|$ is the maximal slope.



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JASON ST
PARK AVE
B M 219
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