Homework 14: Tangent lines and planes

This homework is due Friday, 10/17 rsp Thursday 10/16.

1 The equation $f(x,y) = x^4y^2 + 5xy^5 = 26$ defines a curve in the xy-plane. Find the tangent line $ax + by = d$ to the curve at $(2,1)$ by computing the gradient $\nabla f(x,y) = \langle a, b\rangle$, and then plugging in the point to get the constant d.

Solution:
Given $f(x,y) = x^4y^2 + 5xy^5 = 26$, we have $f_x(x,y) = 4x^3y^2 + 5y^5$, $f_y(x,y) = 2yx^4 + 25xy^4$. Thus the gradient is $\nabla f(x,y) = \langle 4x^3y + 5y^5, 2yx^4 + 25xy^4\rangle$. Plugging in $(2,1)$ gives us $\nabla f = \langle 37,\ 82\rangle$. To find the equation of the line, we just need to solve for the constant: $37x + 82y = d \Rightarrow d = 156$.

2 a) Find an equation of the tangent plane to the surface $z = 3(x-1)^2 + 2(y+3)^2 + 7$ at the point $(2,-2,12)$.
b) Find an equation of the tangent plane to the surface

$$z = \log(x - 2y)$$

(with log = ln as natural log as usual) at the point $(3,1,0)$.

Solution:
a) $3(x-1)^2+2(y+3)^2-z=-7 \Rightarrow f_x(x,y)=6(x-1)$, $f_y(x,y)=4(y+3)$, so $f_x(2,-2)=6$ and $f_y(2,-2)=4$. The tangent plane is $6x+4y-z=d$ where d is a constant. The constant is obtained by plugging in the point. It gives $d=-8$. The equation is $6x+4y-z=-8$.
b) First write it as $\log(x-2y)-z=0$. The gradient is $\langle 1/(x-2y), -2/(x-2y), -1\rangle$. If we plug in the point, we get $1, -2, -1$. The equation is $x-2y-z=d$ where d is a constant Plugging in the point gives $d=1$. The equation is $\boxed{x-2y-z=1}$.

3 a) Find an equation of the tangent plane to the parametric surface

$$\vec{r}(u,v)=\langle u^2, v^2, uv\rangle$$

at the point $(u,v)=(1,1)$.
b) The surface satisfies the equation $xy-z^2=0$. Find the tangent plane to this surface at the same point $(x,y,z)=(1,1,1)$ by computing the gradient.

Solution:
a) $\vec{r}(u,v) = \langle u^2, v^2, uv\rangle \Rightarrow \vec{r}(1,1) = (1,1,1)$. $\vec{r}_u = \langle 2u, 0, v\rangle$; $\vec{r}_v = \langle 0, 2v, u\rangle$, so that a normal vector to the surface at the point $(1,1,1)$ is $\vec{r}_u(1,1) \times \vec{r}_v(1,1) = \langle 2,0,1\rangle \times \langle 0,2,1\rangle = \langle -2,-2,4\rangle$. The tangent plane at the point $(1,1,1)$ is $\boxed{-2x - 2y + 4z = d}$ where the constant d is obtained by plugging in the point $(1,1,1)$ and is 0. The equation is $\boxed{-2x - 2y + 4z = 0}$.
b) We have $\nabla f(x,y,z) = \langle y, x, -2z\rangle$. At the point $(1,1,1)$ we get $\langle 1,1,-2\rangle$ which is parallel to the vector obtained in a). The equation is again of the form $x + y - 2z = d$, where $d = 0$. Again $\boxed{x + y - 2z = 0}$.

4 Find an equation of the tangent plane and the normal line to the surface $x - z - 4\arctan(yz) = 0$ through the point $(1+\pi, 1, 1)$.

Solution:
Let $f(x,y,z) = x - z - 4\arctan(yz)$. Then $x - z = 4\arctan(yz)$ is the level surface $f(x,y,z) = 0$, and $\nabla f(x,y,z) = \langle 1, -\frac{4z}{1+y^2z^2}, -1 - \frac{4y}{1+y^2z^2}\rangle$.
a) Tangent plane: $\nabla f(1+\pi, 1, 1) = \langle 1, -2, -3\rangle$ and an equation of the tangent plane $x - 2y - 3z = d$, where d is the constant obtained by plugging in the point. This is $\boxed{x - 2y - 3z = \pi - 4}$.
b) **Normal line**: The normal line has direction $\langle 1, -2, -3\rangle$, so that the parametrisation is $\boxed{\vec{r}(t) = \langle 1 + \pi + t, 1 - 2t, 1 - 3t\rangle}$.

5 Show that the ellipsoid $6x^2 + 4y^2 + 2z^2 = 18$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1,1,2)$, meaning that they have the same tangent

plane at that point.

Solution:
First note that the point $(1,1,2)$ is on both surfaces. The ellipsoid is a level surface of $f(x,y,z) = 3x^2 + 2y^2 + z^2$ and $\nabla f(x,y,z) = \langle 6x, 4y, 2z\rangle$. A normal vector to the surface at $(1,1,2)$ is $\nabla f(1,1,2) = \langle 6,4,4\rangle$ and an equation of the tangent plane there is $6(x-1)+4(y-1)+4(z-2) = 0$ or $6x+4y+4z = 18$ or $3x + 2y + 2z = 9$.
The sphere is a level surface of $g(x,y,z) = x^2+y^2+z^2-8x-6y-8z+24 = 0$ and $\nabla g(x,y,z) = \langle 2x-8, 2y-6, 2z-8\rangle$. A normal vector to the sphere at $(1,1,2)$ is $\nabla g(1,1,2) = \langle -6,-4,-4\rangle$ and the tangent plane is $6x + 4y + 4z = 18$.

Main definitions

The **gradient** in two dimensions is $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$. In three dimensions, it is $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$.

From the chain rule, we can deduce:

Theorem: Gradients are orthogonal to level curves and level surfaces.

The tangent line through (x_0, y_0) to a level curve $f(x,y) = c$ is $ax + by = d$, where $\nabla f(x_0, y_0) = \langle a, b\rangle$ and d is obtained by plugging in the point.

The tangent plane through (x_0, y_0, z_0) to a level surface $f(x, y, z) = C$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = \langle a, b, c \rangle$ and d is obtained by plugging in the point.

We can compute tangent planes also for parametrized surfaces $\vec{r}(u, v)$ because the vectors $\vec{r}_u, \vec{r}_v$ are velocity vectors of grid curves and so tangent to the surface. Get the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle a, b, c \rangle$ and then get $ax + by + cz = d$, where d is obtained by plugging in the point $\vec{r}(u_0, v_0)$. Here is how to plot parametric surfaces:

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ParametricPlot3D[{u,v^2,u v},{u,0,1},{v,0,1}]
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