

## Homework 13: Chain Rule

This homework is due Wednesday, 10/14 (Monday 10/12 is Columbus day and no class) resp Tuesday 10/13.

- 1 Use the chain rule to find the derivative  $dz/dt$  for

$$z = \cos(x + 7y) ,$$

where  $x(t) = 5t^4$  and  $y(t) = 1/t$ .

**Solution:**

By the chain rule,

$$\begin{aligned} dz/dt &= -\sin(x + 7y) \cdot 20t^3 - 7 \sin(x + 7y) \cdot (-1/t^2), \\ &= -\sin(5t^4 + 7/t) \cdot 20t^3 - 7 \sin(5t^4 + 7/t) \cdot (-1/t^2). \end{aligned}$$

- 2 Find  $dy/dx = y'(x)$  if  $x, y$  are related by

$$\sin(x) + \cos(y) = \sin(x) \cos(y) .$$

**Solution:**

Implicit differentiation with respect to  $x$  yields:

$$\cos(x) - \sin(y)y' = \cos(x) \cos(y) - \sin(x) \sin(y)y'$$

Therefore,

$$y' = \frac{\cos(x) \cos(y) - \cos(x)}{\sin(x) \sin(y) - \sin(y)} .$$

- 3 Find  $z_x$  and  $z_y$  for  $yz = \log(x + z)$ , where  $\log = \ln$  is the natural log.

**Solution:**

Implicit differentiation with respect to  $x$  gives us:

$$yz_x = \frac{1 + z_x}{x + z}$$

Hence  $yz_x(x + z) = 1 + z_x$  or

$$z_x = \frac{1}{y(x + z) - 1}.$$

Implicit differentiation with respect to  $y$  gives us:

$$z + yz_y = \frac{z_y}{x + z}$$

Hence  $(x + z)z + (x + z)yz_y = z_y$  or

$$z_y = \frac{(x + z)z}{1 - (x + z)y}.$$

- 4 The radius of a right circular cone is increasing at a rate of 1.8 while its height is decreasing at a rate of 2.5. At what rate is the volume of the cone changing when the radius is 120 and the height is 140.

**Solution:**

Denote the radius and height of the cone by  $r$  and  $h$  respectively. Then,  $\frac{\partial r}{\partial t} = 1.8$  and  $\frac{\partial h}{\partial t} = -2.5$ . The volume  $V(r, h)$  is given by  $\pi r^2 h/3$ . By the chain rule,

$$\frac{\partial V}{\partial t} = \pi 2rh/3 \cdot 1.8 + \pi r^2/3 \cdot (-2.5).$$

Evaluating at  $r = 120$  and  $h = 140$ ,

$$\frac{\partial V}{\partial t}(120, 140) = \pi 2rh/3 \cdot 1.8 - \pi r^2/3 \cdot 2.5 = 8160\pi.$$

- 5 The Voltage  $V$  in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance  $R$  is slowly increase as the resistor heats up. Use **Ohm's Law**,  $V = IR$ , to find how the current  $I$  is changing at the moment when  $R = 400$ ,  $I = 0.08$   $dV/dt = -0.01$  and  $dR/dt = 0.03$ .

**Solution:**

By the chain rule,

$$-0.01 = \frac{\partial V}{\partial t} = \frac{\partial I}{\partial t} R + I \frac{\partial R}{\partial t} = 400 \frac{\partial I}{\partial t} + 0.08 \cdot 0.03.$$

Thus,  $\frac{\partial I}{\partial t} \approx -0.000031$ .

## Main definitions

Define the **gradient**  $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$  or  $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ .

The **multivariable chain rule** is

$$\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t).$$

When written out in two dimensions, this is

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

**Example:** a bug walks on  $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$  on a plane with temperature  $f(x, y) = x^2 + 5y$ . Find the temperature change  $d/dt f(\vec{r}(t))$  at  $(1, 0)$ . **Solution:** either compose  $f(\vec{r}(t)) = \cos^2(t) + 5 \sin(t)$  and differentiate at  $t = 0$  to get  $d/dt f(\vec{r}(t)) = 5 \cos(0) = 5$ . Or then find  $\vec{r}'(0) = \langle 0, 1 \rangle$  and the gradient  $\nabla f(x, y) = \langle 2x, 5 \rangle$  which is  $\langle 0, 5 \rangle$  at  $(1, 0)$ . The chain rule assures that the dot product is the same.

We can use the chain rule for implicit differentiation

**Implicit differentiation:** If  $f(x, y) = c$  is a curve, we can compute  $y' = -f_x/f_y$ .

In three dimensions, the **implicit differentiation formulas** derived from the chain rule are:

$$z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$$

$$z_y(x, y) = -f_y(x, y, z)/f_z(x, y, z)$$