Homework 12: Linearization

This homework is due Friday, 10/9 rsp Tuesday 10/13.

1 a) Estimate $999990000^{1/3}$ without calculator by linearizing $f(x) = x^{1/3}$ at $x = 1000000000$. Compare with the actual value obtained with a calculator.
b) Find the linearization $L(x, y)$ of the function $f(x, y) = x^6y^7$, at the point $(1, 1)$. Compare $L(1.01, 0.999)$ with $f(1.01, 0.999)$.

Solution:
a) $L(x) = 1000 + (x - 1000)/(3 * 1000^2) = 999.997$. The real number is 999.996666.
b) $L(x) = 1 + 6/100 - 7/1000 = 1.053$. The real value is 1.05411.

2 Find the linear approximation $L(x, y)$ of the function

$$f(x, y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2, 1)$ and use it to estimate $f(1.95, 1.04)$.

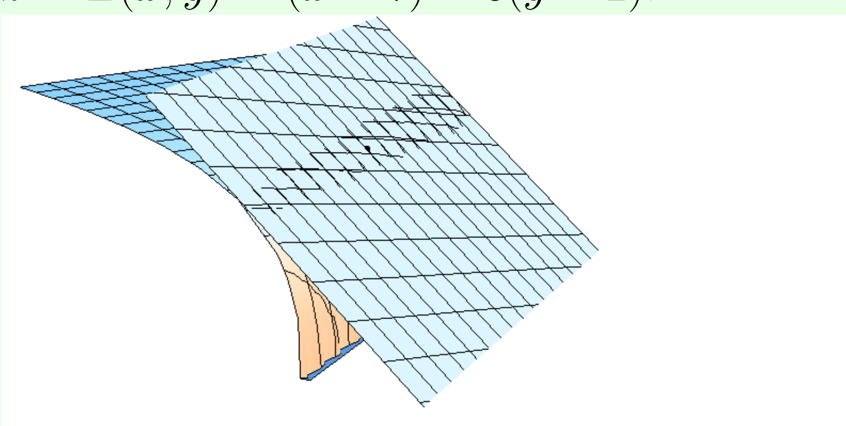
Solution:
$f(2, 1) = 1$, $\nabla(x, y) = (-2x, -10y)/(2f(x, y) = (-x, -5y) = (-2, -5)$. $L(x, y) = f(2, 1) + \nabla f(2, 1)(x - 2, y) = 1 + (-2, -5)(x - 2, y) = 1 - x + (1 - \pi)(y - 1)$ so that
$L(1.95, 1.04) = f(2, 1) + \nabla f(2, 1)(-0.05, 0.04) = 1 + (-2, -5)(-0.05, 0.04) = 1 + 0.10 - 0.2 = 0.90$ which can be compared with the actual value $f(1.95, 1.04) = 0.88888$. We could compute the linearlization without taking square roots.

3 You know that the linearization of $f(x,y) = \sqrt{y+\cos^2 x}$ at a point (x_0, y_0) is $1 + \frac{1}{2}y$. Find (x_0, y_0).

Solution:
We know that $f_y(x_0, y_0) = 1/2$ and $f_x(x_0, y_0) = 0$. Lets restrict x_0 to $(0, 2\pi)$. Now we know $\sqrt{y+\cos^2(x)} = 1$ and $\cos(x_0)\sin(x_0) = 0$ and $f(x_0, y_0) = 1$. The only possibility is $x_0 = 0, \pi$, and $y = 0$.

4 Find the linear approximation $L(x,y)$ of the function $f(x,y) = \ln(x-3y)$ at $(7,2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing $z = f(x,y)$ and the plane $z = L(x,y)$, which we will later call tangent plane.

Solution:
First, $f(7,2) = \ln 1 = 0$. Second, $\frac{\partial f}{\partial x} = \frac{1}{x-3y}$, so $\frac{\partial f}{\partial x}(7,2) = 1$; $\frac{\partial f}{\partial y} = \frac{-3}{x-3y}$, so $\frac{\partial f}{\partial y} = -3$. It follows that the linearization is $z = L(x,y) = (x-7) - 3(y-2)$.



$f(6.9, 2.06) \approx L(6.9, 2.06) = -0.28$. The real error is -0.328504.

5 If $z = x^2 - xy + 3y^2$ and (x,y) changes from $(3,-1)$ to $(2.96, -0.95)$, compare the values of $f(x,y) - f(x_0,y_0)$ and $L(x,y) - L(x_0,y_0)$.

(The later value would be call a differential, but we do not use this expression).

Solution:
We compute: $\frac{\partial z}{\partial x} = 2x - y$ and $\frac{\partial z}{\partial y} = -x + 6y$. It follows that linearization L is $15 + 7x - 9y$. Thus,

$$f(x, y) - f(x_0, y_0) \approx 0.7189$$

while

$$L(x, y) - L(x_0, y_0) \approx 0.73,$$

so the two values are approximately equal.

Main definitions:

The **linear approximation** of a function $f(x)$ at a point a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \ .$$

Example: Because $f(x) = \sqrt{x}$ has at the point $x_0 = 100$ the linearization $L(x) = f(x_0) + f'(x_0)(x - x_0) = 10 + (x - x_0)/20$, we can estimate $f(103) = \sqrt{103}$ as $L(103) = 10 + 3/20 = 10.15$ which is pretty close to the real value 10.1489.

The **linear approximation** of $f(x, y)$ at (a, b) is the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \ .$$

Example. Take $f(x, y) = \sqrt{x^3 y}$ and linearize near $(x_0, y_0) = (3, 3)$. We have $f_x(x, y) = 3x^2/(2\sqrt{x^3 y})$ and $f_y(x, y) = x^3/(2\sqrt{x^3 y})$ so that $f_x(3, 3) = 9/2$ and $f_y(3, 3) = 3/2$. linearization $L(x, y) = 9 + (9/2)(x - 3) + (3/2)(y - 3)$. We can estimate $\sqrt{2.999^3 * 3.00002}$ as $9 + (9/2)(-0.001) + (3/2)0.00002 = 8.99553$, which is $3.6 * 10^{-7}$ close to the real value.

The **linear approximation** of a function $f(x, y, z)$ at (a, b, c) is $L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$.

"Differentials" is outdated terminology which is used in many different ways. It informally refers to the value of $L(x, y) - L(x_0, y_0)$. While tangent lines and tangent planes are level curves or level surfaces of L. We will have a special lecture on this and compute them more efficiently.