Homework 10: Partial derivatives

This homework is due Monday, 10/5 rsp Tuesday 10/6.

1 If $f(x,y) = \sqrt{4 - x^2 + 4y^2}$, find $f_x(1,2)$ and $f_y(1,2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

Solution:
$f(x,y) = \sqrt{4 - x^2 - 4y^2} \Rightarrow f_x(x,y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x,y) = -4y(4 - x^2 - 4y^2)^{-1/2}$. $-1/\sqrt{19}$. $\sqrt{8}/\sqrt{19}$.

2 Find the partial derivatives $f_x(x,y), f_y(x,y)$ of the function $f(x,y) = 5x^y$ at the point $(1,2)$.

Solution:
$f(x,y) = 5x^y \Rightarrow f_x(x,y) = 5yx^{y-1}, f_y(x,y) = 5x^y \ln x$

3 Find the first partial derivatives $f_x(x,y), f_y(x,y)$ of the function

$$f(x,y) = \int_y^x \sin(t^2)\ dt\ .$$

Solution:
$f(x,y) = \int_y^x \sin(t^2)\ dt \Rightarrow f_x(x,y) = \frac{\partial}{\partial x}\int_y^x \sin(t^2)\ dt =$ $\sin(x^2)$ by the Fundamental Theorem of Calculus. $f_y(x,y) = \frac{\partial}{\partial y}\int_y^x \sin(t^2)\ dt = -\frac{\partial}{\partial y}\sin(t^2) = -\sin(y^2)$.

4 Verify that the function

$$u(x,t) = e^{-\alpha^2 k^2 t}\sin(kx)$$

is a solution of the heat conduction equation $u_t(x,t) = \alpha^2 u_{xx}(x,t)$. Here k, α are constants.

Solution:
$u = e^{-\alpha^2 k^2 t}\sin(kx) \Rightarrow u_x = ke^{-\alpha^2 k^2 t}\cos(kx), u_{xx} = -k^2 e^{-\alpha^2 k^2 t}\sin(kx)$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin(kx)$. Thus $\alpha^2 u_{xx} = u_t$

5 Verify that the function

$$u(x,y,z) = 1/\sqrt{x^2+y^2+z^2}$$

is a solution of the three dimensional Laplace equation $u_{xx}+u_{yy}+u_{zz} = 0$.

Solution:
$u = 1/\sqrt{x^2+y^2+z^2} \Rightarrow u_x = (-\frac{1}{2})(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and $u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x(-\frac{3}{2})(x^2+y^2+z^2)^{-5/2}(2x) = \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$. By symmetry, we can see that $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$. Thus $u_{xx} + u_{yy} + u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

Main definitions

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly. We also write $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. and $f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**. Here are examples we are going to look at next time:

1 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound.

2 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic.

3 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane.

4 The **advection equation** $f_t = f_x$ is used to model transport in a wire.

5 The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics.

6 The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break.