Homework 1: Geometry and Distance

This homework is due Friday, 9/11 respectively Tuesday 9/15 at the beginning of class.

1 a) Find the distance a from $P = (3, 4, -1)$ to to the xy-plane.
b) Find the distance b from P to the z-axes.
c) Find the distance c from P to the origin $O = (0, 0, 0)$.
d) What is $a^2 + b^2 - c^2$?

Solution:
a) The distance to the xy-plane is 1. b) The distance to the z axes is $\sqrt{3^2 + 4^2} = 5$. c) The distance to the origin is $\sqrt{26}$. d) This is zero. This is true for all points, if a, b, c are defined as such.

2 a) Find its center and radius of the sphere S:

$$x^2 + y^2 + z^2 - 10x - 8y - 2z - 7 = 0 .$$

b) Find the distance from the center of S to the sphere $x^2 + y^2 + z^2 = 400$.
c) Find the minimal distance between the spheres. This is the minimal distance between two points where each is in one sphere.

Solution:
a) Complete the square:

$$\begin{aligned} x^2 - 10x + 25 + y^2 - 8y + 16 + z^2 - 2z + 1 &= 25 + 16 + 1 + 7 \\ (x-5)^2 + (y-4)^2 + (z-1)^2 &= 7^2 . \end{aligned}$$

Thus, the center of the sphere is at $(5, 4, 1)$ and the radius is 7.
b) The sphere S has distance $\sqrt{5^2 + 4^2 + 1^2} = \sqrt{42}$ to the origin. Subtract this from 20. The answer is $20 - \sqrt{5^2 + 4^2 + 1^2}$.
c) Now we also have to subtract the radius of the smaller sphere. The answer is $13 - \sqrt{5^2 + 4^2 + 1}$.

3 a) Find an equation of the largest sphere with center $(8, 11, 9)$ that is contained in the first octant $\{x > 0, y > 0, z > 0\}$.
b) Find the equation for the sphere centered at $(6, 10, 8)$ which passes through the center $(8, 11, 9)$ of the sphere in a).

Solution:
a) The closest coordinate plane is the yz plane. It has distance 8, so the radius of the largest sphere must be 8. The equation is $(x - 8)^2 + (y - 11)^2 + (z - 9)^2 = 8^2$.
b) The distance between $(6, 10, 8)$ and $(8, 11, 9)$ is $\sqrt{6}$. Thus, the sphere's equation is $(x - 6)^2 + (y - 10)^2 + (z - 8)^2 = 6$.

4 a) What is the surface $(x-1)^2+(z+2)^2 = 16$ in three dimensional space R^3.
b) What is the surface $x^2 = z^2$ in three dimensional space R^3.
c) What is the intersection of the two surfaces? Draw both!

Solution:
a) This is a cylinder centered around the axis $x = 1, z = -2$ with radius 4.
b) This is a union of two planes $x = z$ and $x = -z$ containing the x-axis. c) The intersection consists of 4 lines.

5 An ant moves on the unit cube bound by the walls $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ from the point $A = (0.4, 0.8, 1)$ to the point $B = (1, 0.8, 0.5)$. Compute the length of the two obvious paths, where one passes over three faces, the other only over two. Which one is shorter? See the figures on the third page.

Solution:
Let's look at the first situation, where we move along three faces and where we draw things in the two dimensional plane. The point B has coordinates $(0.8, 0.5)$ and the point A has coordinates $(1.6, 1.2)$. The distance between them is $\sqrt{0.8^2 + 0.7^2} = 1.06$.
In the second case, we have the points $(0.8, 0.5)$ and $(0.8, 1.6)$. The distance is 1.1. The first distance is a tiny bit shorter.

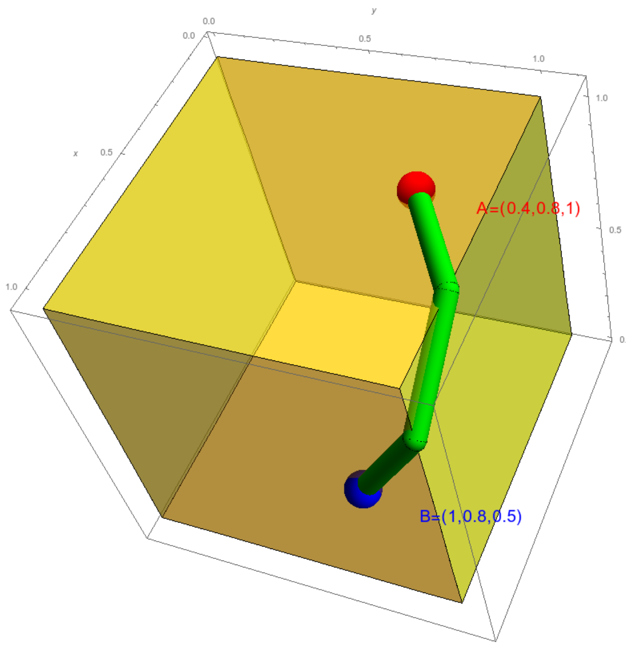
Main definitions

A point in the **plane** has **coordinates** $P = (x, y)$. A point in **space** has coordinates $P = (x, y, z)$. The coordinate signs define 4 **quadrants** in the plane and 8 **octants** in space. These regions by intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$, and are separated by **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$ which intersect in **coordinate axes** like the z-axes $\{y = 0, x = 0\}$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$. The distance between a point P and a geometric object S like a line or plane or sphere is the minimal distance $d(P, Q)$ which can be achieved among all points Q located on S.

A **circle** of radius r centered at $P = (a, b)$ is the collection of points in the plane which have distance r from P. A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

We **complete the square** of $x^2 + bx + c = 0$ by adding$(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives $x = -b/2 \pm \sqrt{(b/2)^2 - c}$. **Example:** Find the center and radius of the circle $x^2 + 8x + y^2 = 9$. **Solution:** Add 16 on both sides to get $x^2 + 8x + 16 + y^2 = 25$ which is $(x+4)^2 + y^2 = 25$, a circle of radius $r = 5$ centered at $(-4, 0)$.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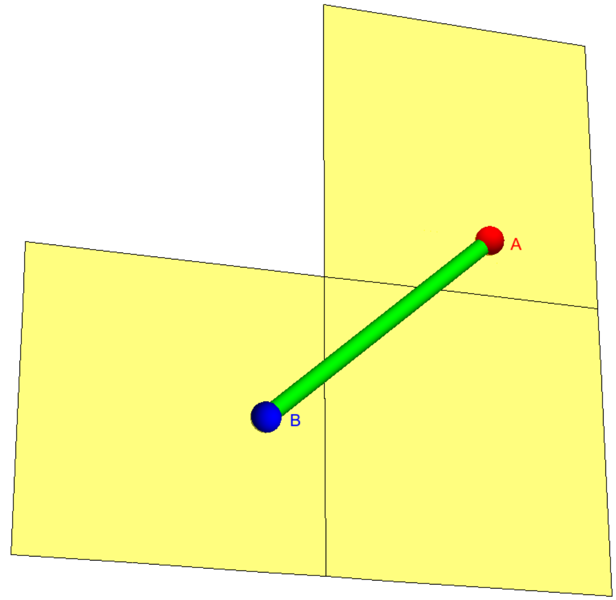
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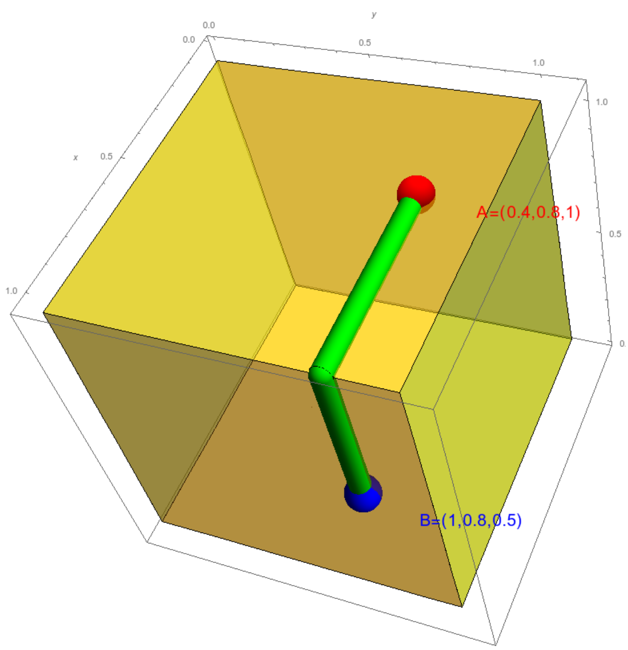
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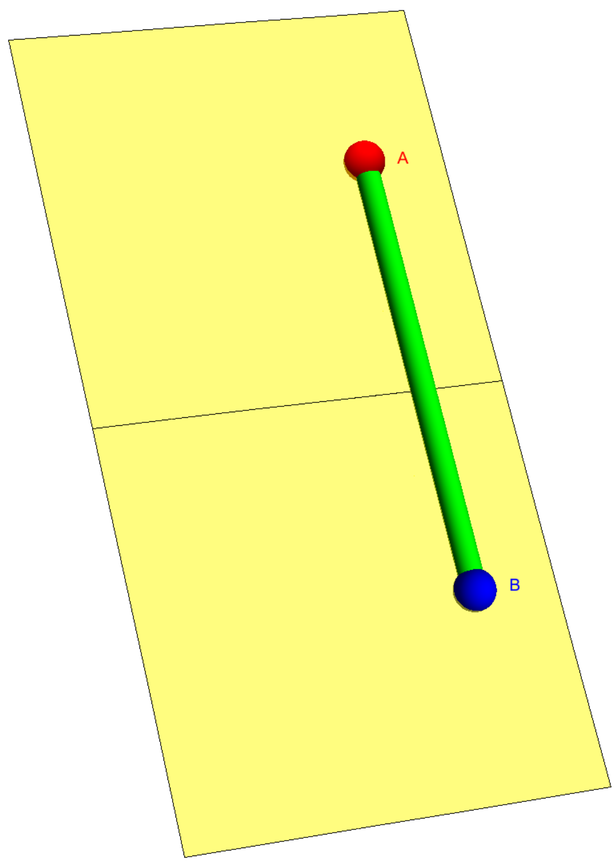
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