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- Start by printing your name in the above box and **check your section** in the box to the left.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader cannot be given credit.

- **Show your work**. Except for problems 1-3, we need to see details of your computation.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) **T** | F   There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v} + \vec{w}$ has length $1/3$.

Solution:
Look at the diagonal of the parallelogram spanned by $\vec{v}$ and $\vec{w}$. It can have any length between 0 and 2.

2) T | **F**   For any three vectors, we have $|(\vec{u} \times \vec{v}) \times \vec{w}| = |(\vec{v} \times \vec{w}) \times \vec{u}|$.

Solution:
For $\vec{u} = \vec{i}, \vec{v} = \vec{j}$ and $\vec{w} = \vec{j}$, the first expression is 1 the last is 0.

3) **T** | F   Denote by $d(P, L)$ the distance from a point P to a line L in space. For any point P and any two lines L, K in space, we have $d(P, L) + d(P, K) \geq d(L, K)$.

Solution:
For any point A on L and any point B on K, we have $d(P, A) + d(P, B) \geq d(A, B)$ by the triangle inequality. This is especially true when A is the point on the line L such that $d(P, A) = d(P, L)$ and when B is the point on the line K such that $d(P, K) = d(P, B)$.

4) **T** | F   For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the relation $|\vec{u} \times (\vec{v} \times \vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$ holds.

Solution:
Use the formula $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\alpha)$ twice.

5) T | **F**   If $\vec{r}(t)$ has speed 1 and curvature 1 everywhere, then $\vec{r}(2t)$ has constant speed 2 and constant curvature $1/2$ everywhere.

Solution:
While the statement is true for the speed, the curvature does not change under reparametrization.

6) **T** | F   If the curvature of a space curve is constant 1 and the speed $|\vec{r}'(t)| = 1$ everywhere, then the acceleration satisfies $|\vec{r}''(t)| = 1$ everywhere.

Solution:
The assumption implies $|\vec{T}| = |\vec{r}'|$ and $\kappa = |\vec{T}'|/|\vec{r}'| = |\vec{T}'| = |\vec{r}'| = |\vec{r}''|$. An other way to think about it: $|r'| = 1$ implies $r'' \cdot r' = 0$ so that $1 = |r'' \times r'|/|r'|^3 = |r''|$.

7) T **F** If a vector field $\vec{F} = \langle P, Q \rangle$ has $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\mathrm{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then $\vec{F}$ must be constant.

Solution:
Given vector field $\vec{F} = \nabla f$, there is a solution f to $\mathrm{div}(\mathrm{grad}(f)) = 0$ like $f(x,y) = x^2 - y^2$. The vector field $\langle 2x, -2y \rangle$ has divergence zero and curl zero.

8) T **F** If the level curve $f(x,y) = 1$ contains both the lines $x = y$ and $x = -y$, then $(0,0)$ must be a critical point for which $D < 0$.

Solution:
It can be a surface with $D = 0$ like $x^4 - y^4$.

9) **T** F The surface $\vec{r}(u,v) = \langle u^3 \cos(v), u^3 \sin(v), u^3 \rangle$ with $v \in [0, 2\pi)$ and $-\infty \le u \le \infty$ is a double cone.

Solution:
$x^2 + y^2 = z^2$.

10) **T** F There is a non-constant function $f(x,y,z)$ of three variables such that $\mathrm{div}(\mathrm{grad}(f)) = f$.

Solution:
For example $f(x,y,z) = e^x$.

11) **T** F If $\mathrm{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\mathrm{div}(\vec{F}) = 0$ everywhere.

Solution:
Indeed, then $\mathrm{div}(\vec{F}) = \mathrm{div}(\mathrm{curl}(\vec{F})) = 0$.

12) **T** | **F**   The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.

Solution:
It is a single cone.

13) **T** | **F**   The tangent plane of $x^3 + y^2 + z^4 = 9$ at $(0, 3, 0)$ is $y = 3$.

Solution:
The gradient is $\langle 0, 6, 0 \rangle$ so that the equation is $6y = d$. Plugging in the point into the equation gives $y = 18$.

14) **T** | **F**   Assume (x_0, y_0) is not a critical point of $f(x, y)$. It is possible that f increases at (x_0, y_0) most rapidly in the direction $\langle 1, 0 \rangle$ and decreases most rapidly in the direction $\langle 4/5, -3/5 \rangle$.

Solution:
The direction of maximal increase is the opposite direction of the direction of maximal decrease.

15) **T** | **F**   Assume $\vec{F}(x, y, z)$ is defined everywhere except on the z-axis and satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere except on the z-axis, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all curves C.

Solution:
You can have $\text{curl}(\vec{F})$ to be singular on the z axes. An example is $\langle -y/(x^2 + y^2), x/(x^2 + y^2), 0 \rangle$.

16) **T** | **F**   A point (x_0, y_0) is an extremum of $f(x, y)$ under the constraint $g(x, y) = 0$. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0, y_0) can not be a local maximum on the constraint curve.

Solution:
The discriminant D has no significance for extremization problems under constraints. Take $x^2 + y^2$ or $-x^2 - y^2$ to get minima or maxima evenso $D > 0$.

17) **T** | **F**   The vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ can be the curl of another vector field $\vec{G}$.

Solution:
We would need that the divergence of $\vec{F}$ is zero.

18) | **T** | F |

If $f(x, y)$ and $g(x, y)$ are two functions and $(2, 3, 3)$ is a critical point of the function $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$, then $(2, 3)$ is a solution of the Lagrange equations for extremizing $f(x, y)$ under the constraint $g(x, y) = 0$.

Solution:
Being a critical point of F means $g(x, y) = 0$ (partial derivative of F with respect to λ is zero) and $f_x = \lambda g_x, f_y = \lambda g_y$ (partial derivatives of F with respect to x, y are zero).

19) | **T** | F |

Assume $(0, 0)$ is a global maximum of $f(x, y)$ on the disc $D = \{x^2 + y^2 \leq 1\}$, then $\int \int_D f(x, y) \, dxdy \leq \pi f(0, 0)$.

Solution:
$\int \int_D f(x, y) \, dxdy \leq \int \int_D f(0, 0) \, dxdy = f(0, 0)\pi$.

20) | T | **F** |

Let C be a curve parametrized by $\vec{r}(t), 0 \leq t \leq 1$ for which the acceleration is constant 1. Then $\int_C \nabla f \cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}''(t)}(f(\vec{r}(t))) \, dt$.
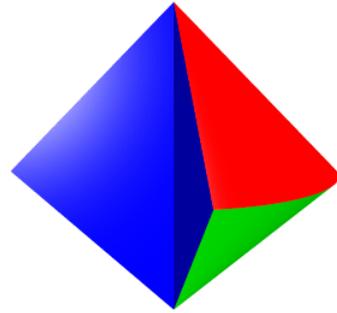
Solution:
It would be true for the velocity, not for the acceleration.
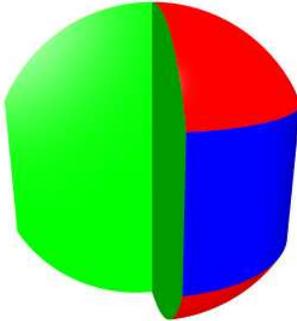

Problem 2) (10 points)

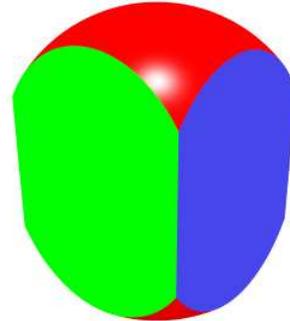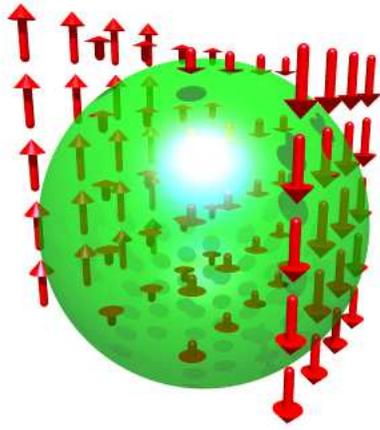a) (4 points) Match the following triple integrals with the regions.
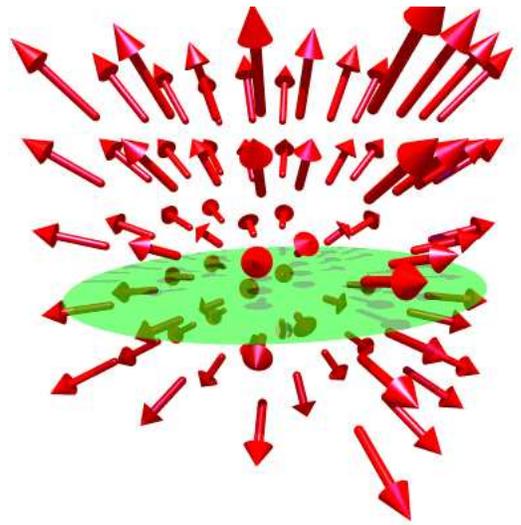
I

II

III
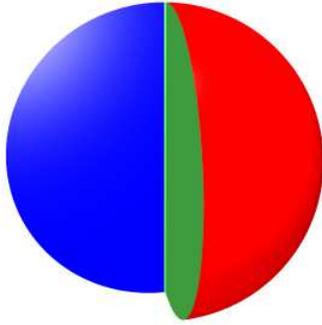
IV

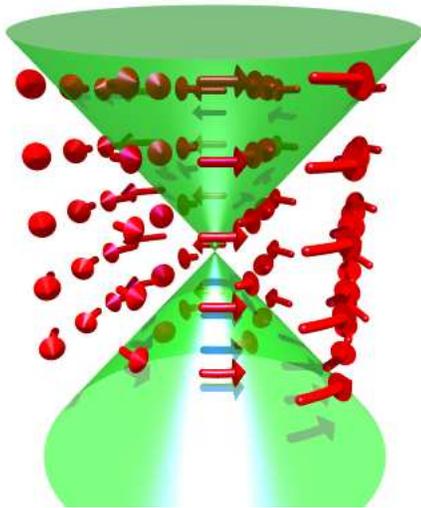| Enter I,II,III,IV here	Equation
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{r-1}^{1-r} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x, y, z) \; dzdydx$

2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.
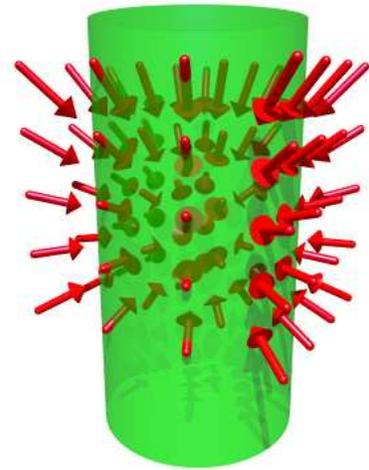
A

B

C

D

| Enter A,B,C,D	Field	Surface	Flux zer
	$\vec{F}(x,y,z) = \langle x, y, z \rangle$	$\vec{r}(u,v) = \langle u, v, 0 \rangle$	
	$\vec{F}(x,y,z) = \langle 0, 0, y \rangle$	$\vec{r}(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$	
	$\vec{F}(x,y,z) = \langle -x, -y, -z \rangle$	$\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$	
	$\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$	$\vec{r}(\theta, z) = \langle z\cos(\theta), z\sin(\theta), z \rangle$	

Solution:

a) III,II,I,IV.

b) B,A,D,C. The flux is zero for B,A,C. In the plane and cone situation the vector field is tangent to the surface. In the sphere case, there is part of the surface where the flux is positive and part of the surface where it is just the opposite. Only for the cylinder, the flux is nonzero. All vectors point inwards.

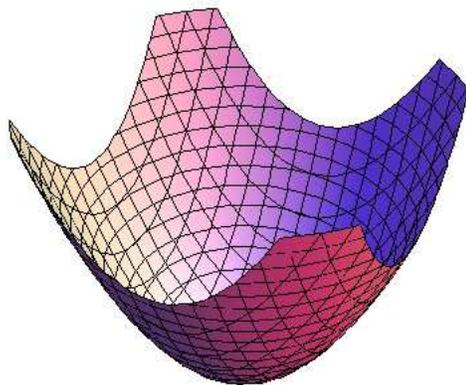---

Problem 3) (10 points)

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a) (6 points) Match the following level surfaces with functions $f(x, y, z)$ and also match the parametrization of part of the surface $f(x, y, z) = 0$.
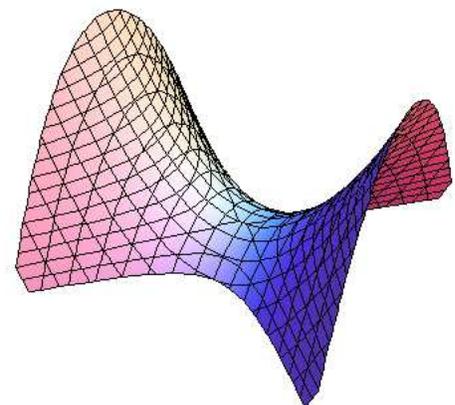

I

II

III
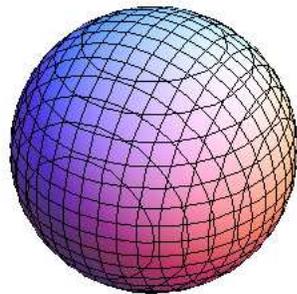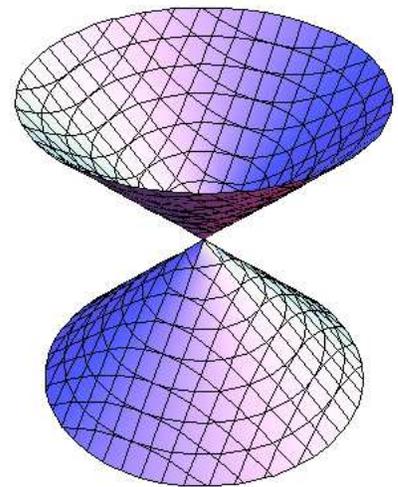
IV

Enter I,II,III,IV	$f(x, y, z) = 0$	Enter I,II,III,IV	parametrization
	$f(x, y, z) = -x^2 + y^2 + z$		$\langle u, v, u^2 - v^2 \rangle$
	$f(x, y, z) = x^2 + y^2 + z^2 - 1$		$\langle u, v, u^2 + v^2 \rangle$
	$f(x, y, z) = -x^2 - y^2 + z$		$\langle u, v, \sqrt{1 - u^2 - v^2} \rangle$
	$f(x, y, z) = -x^2 - y^2 + z^2$		$\langle s\cos(t), s\sin(t), s \rangle$

b) (2 points) We know that
$\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$,
$\vec{r}(0) = \langle 2, 3, 4 \rangle$ and
$\vec{r}\,'(0) = \langle 0, 1, 1 \rangle$.
The expression $\langle \cos(t)+1, \sin(t)+3, t+4 \rangle$ is equal to:

Check which applies	result
	the velocity $\vec{r}\,'(t)$
	the position $\vec{r}(t)$
	the curvature $\kappa(\vec{r}(t))$
	the unit tangent vector $\vec{T}(t)$

c) (2 points) What is the name of
the partial differential equation
$\text{div}(\text{grad}(f)) = 0$ for $f(x, y)$?

Check which applies	PDE
	Transport equation
	Wave equation
	Heat equation
	Laplace equation

Solution:
a) IV,I,III,II and IV,III,I,II
b) It is the position. (The problem had a typo with $\langle 0, 1, 0 \rangle$. We did not take any points off in b).)
c) This is the Laplace equation.

Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2 + y^2 + (z-6)^2 = 1$ and the cylinder of radius 2 around the line $x = y = z$.

Solution:
The points $A = (0, 0, 0)$ and $B = (1, 1, 1)$ are on the line so that the perametrization of the line is $\vec{r}(t) = A + t\vec{v}$ with $\vec{v} = \vec{AB} = \langle 1, 1, 1 \rangle$. We first compute the distance between the point $P = (4, 0, 6)$, the center of the sphere, and the line. This distance is

$$d = \frac{|\vec{PA} \times \vec{v}|}{|\vec{v}|} = \frac{|\langle 4, 0, 6 \rangle \times \langle 1, 1, 1 \rangle|}{|\langle 1, 1, 1 \rangle|}$$

$= \frac{|\langle -6, 2, 4 \rangle|}{\sqrt{3}}$. We have to subtract from this the radius of the sphere and the radius of the cylinder, so that the result is now $\boxed{\sqrt{56/3} - 3}$.

a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface S at $(1, 1, 2)$.

Solution:
a) The gradient of $f(x, y, z) = 4xy - z^2$ is $\nabla f(x, y, z) = \langle 4y, 4x, -2z \rangle$. The gradient at $\langle 1, 1, 2 \rangle$ is $\langle 4, 4, -4 \rangle$. The equation of the tangent plane is $4x + 4y - 4z = d$, where the constant $d = 0$ can be obtained by plugging in the point $(x, y, z) = (1, 1, 2)$. The plane is $\boxed{x + y - z = 0}$.

b) Since $f(1, 1, 2) = 0$, we get with the linearization $L(x, y, z) = 0 + 4(x - 1) + 4(y - 1) - 4(z - 2) = 0.004 - 0.04 - 0.004 = \boxed{-0.04}$

c) $\vec{r}(t) = \langle 1, 1, 2 \rangle + t \langle 4, 4, -4 \rangle$ which is $\boxed{\vec{r}(t) = \langle 1 + 4t, 1 + 4t, 2 - 4t \rangle}$.

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.
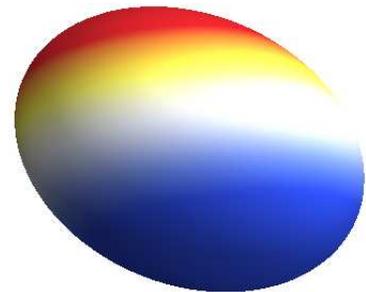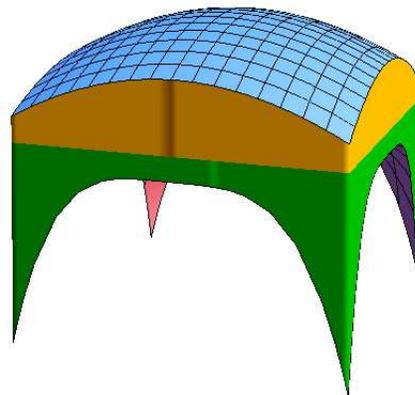
Solution:
The Lagrange equations are

$$5 = \lambda 10x$$
$$-2 = \lambda 2y$$
$$9 = \lambda 6z$$
$$5x^2 + y^2 + 3z^2 = 9$$

Dividing the first equation by the second we get $y = -2x$. If we divide the first by the third, we get relation $z == 3x$. Plugging this into the third 4th equation gives $36x^2 = 9$ or $x = \pm 1/2$. The critical points are $(1/2, -1, 3/2)$ and $(-1/2, 1, -3/2)$. We evaluate f at these two points to get $f(1/2, -1, 3/2) = 768, f(-1/2, 1, -3/2) = 732$. The point $\boxed{(1/2, -1, 3/2)}$ is the maximum.

Problem 7) (10 points)

The thickness of the region enclosed by the two graphs $f_1(x, y) = 10 - 2x^2 - 2y^2$ and $f_2(x, y) = -x^4 - y^4 - 2$ is denoted by $f(x, y) = f_1(x, y) - f_2(x, y)$. Classify all critical points of f and find the global minimal thickness.

To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

Solution:
The function to extremize is $f(x,y) = 12 + x^4 + y^4 - 2x^2 - 2y^2$ Its gradient is $\nabla f(x,y) = \langle 4x^3 - 4x, 4y^3 - y \rangle$. This gradient is equal to $\langle 0,0 \rangle$ if $x \in \{0, 1, -1\}$ and $y \in \{0, 1, -1\}$. There are 9 critical points. Now we proceeed and use the second derivative test. We compute the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 4)(12y^2 - 4)$ and $f_{xx} = 12x^2 - 4$. D is negative if exactly one of the x, y is zero. Otherwise, it is positive. f_{xx} is negative

x	y	D	$f_x x$	Type	f(x,y)=
-1	-1	64	8	minimum	10
-1	0	-32	8	saddle	11
-1	1	64	8	minimum	10
0	-1	-32	-4	saddle	11
0	0	16	-4	maximum	12
0	1	-32	-4	saddle	11
1	-1	64	8	minimum	10
1	0	-32	8	saddle	11
1	1	64	8	minimum	10

if $x = 0$.   The minimal value 10 occurs at 4

places. These are $\boxed{(-1,-1), (-1,1), (1,-1), (1,1)}$. These are local minima. But they are also global minima because $f(x,y) = (x^2 - 1)^2 + (y^2 + 1)^2 + 10$ is always positive and goes to infinity for $(x,y) \to \infty$.

Problem 8) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.
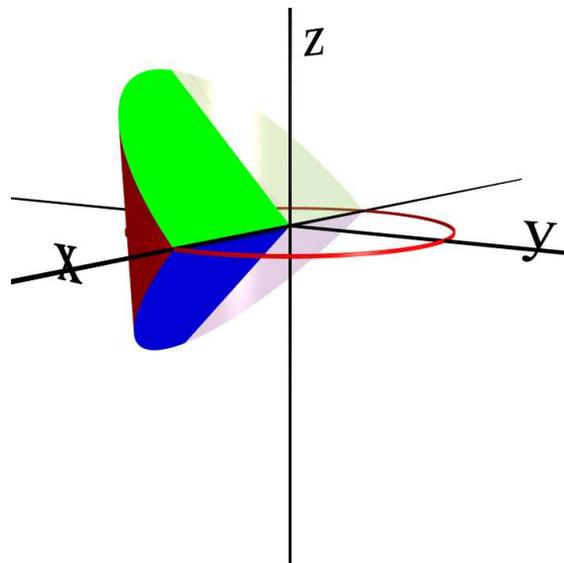
Solution:
We use cylindrical coordinates. The base region in the xy plane is the forth quadrant. Its roof is $z = -y$, its floor is $z = y$. We have to integrate $f(x,y) = -2y = -2r\sin(\theta)$ over the fourth quadrant and get:

$$\int_0^1 \int_{3\pi/2}^{2\pi} -2r\sin(\theta) r \, d\theta dr = 2/3 \; .$$

The volume of the cheese is $\boxed{2/3}$.

Problem 9) (10 points)

Compute the surface area of the **Tsai** surface which is parametrized by

$$\vec{r}(u,v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle \; ,$$

where $0 \le u \le 1$ and $u^{1/4} \le v \le 1$.

Solution:
We have $\vec{r}_u(u,v) = \langle 3, 4, 0 \rangle$ and $\vec{r}_v(u,v) = \langle 2, 1, v^{5/2} \rangle$, so that

$$\vec{r}_u \times \vec{r}_v = \langle 4v^{5/2}, -3v^{5/2}, -5 \rangle \; .$$

Its magnitude is $|\vec{r}_u \times \vec{r}_v| = \sqrt{25v^5 + 25} = 5\sqrt{v^5 + 1}$. The surface area is

$$\int\int |\vec{r}_u \times \vec{r}_v| \, dvdu = \int_0^1 \int_{u^{1/4}}^1 5\sqrt{v^5 + 1} \, dvdu \; .$$

Since this can not be solved directly, we have to change the order of integration:
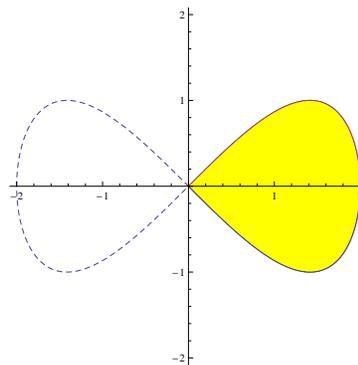
$$\int_0^1 \int_0^{v^4} 5\sqrt{v^5 + 1} \, dudv = \int_0^1 5v^4\sqrt{v^5 + 1} \, dv = \frac{2}{3}(2\sqrt{2} - 1) \; .$$

The result is $\boxed{\frac{2}{3}(2\sqrt{2} - 1)}$.

Problem 10) (10 points)

Find the area $\int \int_R 1 \, dxdy$ of the region R inside the right leaf of the **Gerono lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2\sin(t), 2\sin(t)\cos(t)\rangle \ .$$

Solution:
We use the vector field $\vec{F}(x, y) = \langle 0, x\rangle$ and use **Greens theorem**. We get

$$\int_0^\pi \langle 0, 2\sin(t)\rangle \cdot \langle 2\cos(t), 4\cos^2(t)-2\rangle = \int_0^\pi 8\sin(t)\cos^2(t) - 4\sin(t) \ dt = \cos^3(t)(8/3) - 4\cos(t)\big|_0^\pi = 8/3$$

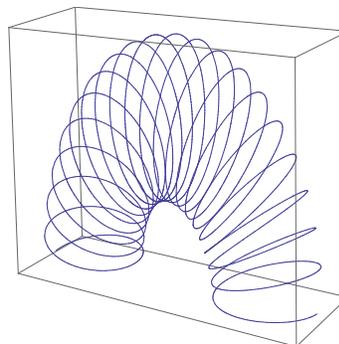The area is $\boxed{8/3}$.

Problem 11) (10 points)

Find the line integral of the vector field

$$\vec{F}(x, y, z) = \langle \cos(x+z), 2yze^{y^2 z} + 7, \cos(x+z) + y^2 e^{y^2 z}\rangle$$

along the **slinky** curve

$$\vec{r}(t) = \langle \sin(40t), (2+\cos(40t))\cos(t), (2+\cos(40t))\sin(t)\rangle$$

with $0 \le t \le \pi$.

Solution:
The vector field $\vec{F}$ is a **gradient field** with $\vec{F}(x, y, z) = \nabla f(x, y, z)$ with $f(x, y, z) = \sin(x+z) + 7y + e^{y^2 z}$. The curve starts at $\vec{r}(0) = \langle 0, 3, 0\rangle$ and ends with $\vec{r}(\pi) = \langle 0, -3, 0\rangle$. The line integral is therefore by the **fundamental theorem of line integrals** $f(r(\pi)) - f(r(0)) = -20 - 22 = \boxed{-42}$. This is the answer to the "ultimate question".
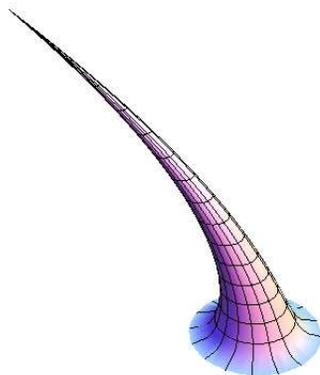

Problem 12) (10 points)

Find the flux integral $\int \int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x, y, z) = \langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz \rangle$$

and S is the **thorn** surface parametrized by

$$\vec{r}(s, t) = \langle (1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s \rangle$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.

Solution:
This problem can be solved in three different ways. 1. solution. The vector field $\vec{F}$ is the sum of the gradient of $f(x, y, z) = \cos(\pi y)e^{2x} + z^2 x$ and $\vec{G}(x, y, z) = \langle 0, x^2\cos(z\pi/2), 0 \rangle$. By Stokes theorem, the flux of $\mathrm{curl}(F) = \pi x^2\sin(\pi z)/2)/2, 0, 2x\cos(\pi z/2)$ is the line integral of $\langle 0, x^2\cos(z\pi/2), 0 \rangle$ along the boundary curve $\vec{r}(t) = \langle \cos(t), \sin(0), 0 \rangle$ which is $\int_0^{2\pi}\langle 0, \cos^2(t), 0 \rangle \cdot \langle \sin(t), \cos(t), 0 \rangle = 0$.
2. Solution. The flux is by Stokes theorem the line integral along the boundary $\vec{r}(t)$ which is by Stokes theorem the flux integral of $\mathrm{curl}(\vec{F}) = \pi x^2\sin(\pi z)/2)/2, 0, 2x\cos(\pi z/2)$ through the disc with that boundary. This flux integral is zero because on the disc $\mathrm{curl}(\vec{F}(x, y, z) = \langle 0, 0, 2x \rangle$ so that the flux is the double integral of $2x$ over the disc which is zero.
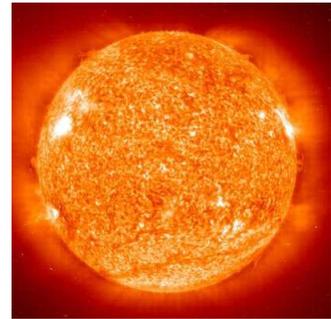3. Solution. The flux through the "thorn" together with the flux through the bottom disc (oriented downwards) closing the surface is zero because $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$. Therefore, the flux through the thorn is the same as the flux through the disc (oriented upwards) which is zero as in the 2. Solution. The result is again $\boxed{0}$.

Problem 13) (10 points)

Assume the vector field

$$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int \int_S \vec{F} \cdot d\vec{S}$.
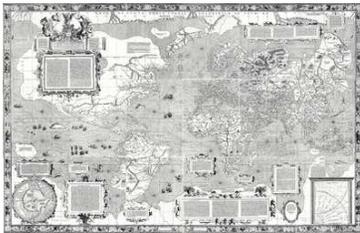

Solution:
The divergence is $15x^2 + 15y^2 + 15z^2$. We integrate this over the sphere to get by the divergence theorem the flux through the surface. To compute the triple integral we use spherical coordinates

$$\int_0^{2\pi} \int_0^\pi \int_0^3 15\rho^2 \rho^2 \sin(\phi) \, d\rho d\phi d\theta .$$

The result is $15(3^5/5)4\pi = 4 \cdot 3^6 \cdot \pi = \boxed{2916\pi}$.

Problem 14) (10 points)

The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u, v) = \langle \cos(u) \cos(\arctan(\sinh(v))), \sin(u) \cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle .$$

a) (3 points) Show that $|\vec{r}(u, v)| = 1$ verifying so that $\vec{r}(u, v)$ parametrizes the unit sphere, if $0 \le u < 2\pi, -\infty < v < \infty$.
b) (3 points) Show that $|\vec{r}_u(u, v)| = |\vec{r}_v(u, v)| = 1/\cosh(v)$ and that $\vec{r}_u(u, v) \cdot \vec{r}_v(u, v) = 0$.
c) (2 points) Use b) to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
d) (2 points) Use $\int 1/\cosh^2(x) \, dx = 2\arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is 4π.

Hint for b): you can use the identity $\cos(\arctan(\sinh(v)) = 1/\cosh(v)$.

Solution:
a) With $w = \pi/2 - \arctan(\sinh(v))$ we can rewrite this as

$$r(\vec{u, w}) = \langle \cos(u)\sin(w), \sin(u)\sin(w), \cos(w)\rangle$$

which is the standard parametrization of the sphere.
b) This is a direct computation using the one-dimensional chain rule.
c) Use the formula $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$ if the vectors a, b are perpendicular.
d) $\int_{-\infty}^{\infty} 1/\cosh^2(x)\, dx = 2$.