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- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) | T | **F** | The parametrization $\vec{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$ describes a cone.

Solution:
It is a cylinder.

2) | **T** | F | The vectors $\vec{v} = \langle 2,1,5 \rangle$ and $\vec{w} = \langle 2,1,-1 \rangle$ are perpendicular.

Solution:
Their dot product is zero.

3) | T | **F** | Let E be a solid region with boundary surface S. If $\iint_S \vec{F} \cdot d\vec{S} = 0$, then $\mathrm{div}(\vec{F})(x,y,z) = 0$ everywhere inside E.

Solution:
There could be a cancellation of divergence like for $\vec{F} = \langle x^2, 0, 0 \rangle$ with the unit sphere S.

4) | **T** | F | If $\mathrm{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F} \cdot d\vec{S} = 0$ for any closed surface S.

Solution:
This follows from the divergence theorem.

5) | T | **F** | If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ is has zero divergence everywhere.

Solution:
The vector field $\vec{F}(x,y,z) = \langle x, 0, 0 \rangle$ is conservative but has divergence 1 everywhere.

6) | **T** | F | If $\vec{F}, \vec{G}$ are two vector fields for which $\vec{F} - \vec{G} = \mathrm{curl}(\vec{H})$, then $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{G} \cdot d\vec{S}$ for any closed surface S.

Solution:
When adding two vector fields, the corresponding flux is added. Since the flux of the curl of a vector field through any closed surface is zero (follows from Stokes or from the divergence theorem), the flux of $\vec{F}$ and $\vec{G}$ is the same.

7) | T | **F** |

The linearization of $f(x, y) = x^2 + y^3 - x$ at $(2, 1)$ is $L(x, y) = 3 + 4(x - 2) + 3(y - 1)$.

Solution:
Almost. It is just the constant which is wrong. The correct linearization is $L(x, y) = 4 + 4(x - 2) + 3(y - 1)$.

8) | **T** | F |

The volume of the solid E is $\int\int_S \langle x, 2x+z, x-y\rangle \cdot d\vec{S}$, where S is the surface of the solid E.

Solution:
Since the divergence of the vector field is 1, this follows from the divergence theorem.

9) | **T** | F |

The vector field $\vec{F}(x, y, z) = \langle x+y, x-y, 3\rangle$ has zero curl and zero divergence everywhere.

Solution:
yes, both are zero.

10) | **T** | F |

If $\vec{F}$ is a vector field, then the flux of the vector field curl(curl(F)) through a sphere $x^2 + y^2 + z^2 = 1$ is zero.

Solution:
By the divergence theorem and using the fact that the vector field is curl(G) for some other vector field $G =$ curl(F).

11) | T | **F** |

If the vector field $\vec{F}$ has zero curl everywhere then the flux of $\vec{F}$ through any closed surface S is zero.

Solution:
It would follows from the divergence theorem if $\vec{F}$ were incompressible. It would also follow that the line integral along any closed curve is zero.

12) T F

The equation div(grad(f)) $= 0$ is an example of a partial differential equation for an unknown function $f(x, y, z)$.

Solution:
If we write it out, it is.

13) T F

The vector $(\vec{i}+\vec{j}) \times (\vec{i}-\vec{j})$ is the zero vector if $\vec{i} = \langle 1, 0, 0 \rangle$ and $\vec{j} = \langle 0, 1, 0 \rangle$.

Solution:
Either compute directly $\langle 1, 1, 0 \rangle \times \langle 1, -1, 0 \rangle = \langle 0, 0, -2 \rangle$ or foil out in your head to get the result is $-2\vec{i} \times \vec{j}$ which is equal to $-2\vec{k}$.

14) T F

If f is maximal under the constraint $g = c$, then the angle between $\nabla f(x, y)$ and $\nabla g(x, y)$ is zero.

Solution:
It is zero or π. The second case is also possible.

15) T F

Let L be the line $x = y, z = 0$ in the plane $\Sigma : z = 0$ and let P be a point. Then $d(P, L) \geq d(P, \Sigma)$.

Solution:
Yes, restricting to a line can make the distance not smaller.

16) T F

The chain rule assures that $\frac{d}{dt} f(\vec{r}\,'(t)) = \nabla f(\vec{r}\,'(t)) \cdot \vec{r}\,''(t)$.

Solution:
Yes. It is just the usual chain rule applied to the parametrization $\vec{r}\,'(t)$.

17) T F

If K is a plane in space and P is a point not on K, there is a unique point Q on K for which the distance $d(P, Q)$ is minimized.

Solution:
This point Q is the projection of the point onto the plane. Every other point has larger distance as we can draw a triangle which has a right angle at Q.

18) | T | F |

If $\vec{B}(t)$ is the bi-normal vector to an ellipse $\vec{r}(t)$ contained in the plane $x + y + z = 1$, then $\vec{B}$ is parallel to $\langle 1, 1, 1 \rangle$.

Solution:
It is always the normal vector

19) | T | F |

The parametrized surface $\vec{r}(u, v) = \langle u, v, u^2 + v^2 \rangle$ is everywhere perpendicular to the vector field $\vec{F}(x, y, z) = \langle x, y, x^2 + y^2 \rangle$.

Solution:
Since we do not see any theoretical reason why this should be true, lets experiment. At the point $\langle 1, 1, 2 \rangle$, the vector field is $\langle 1, 1, 2 \rangle$ and the normal vector to the surface is $\langle 1, 0, 2 \rangle \times \langle 0, 1, 2 \rangle = \langle -2, -2, 1 \rangle$. The dot product is not zero.

20) | T | F |

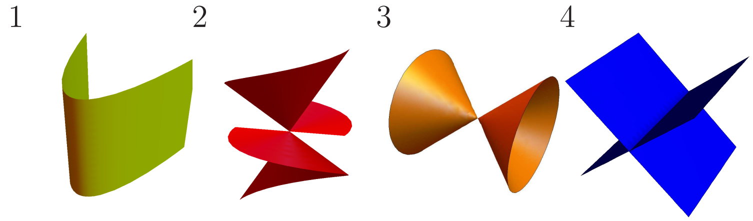
Assume $\vec{r}(t)$ is a flow line of a vector field $\vec{F} = \nabla f$. Then $\vec{r}\,'(t) = \vec{0}$ if $\vec{r}(t)$ is located at a critical point of f.

Solution:
By definition $\vec{r}'(t) = \vec{F}(\vec{r}(t)) = \nabla f(\vec{r}(t))$. At a critical point, this is zero.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), uv\rangle$	
$\vec{r}(u,v) = \langle u, u^2, v\rangle$	
$x^2 - y^2 = z^2$	



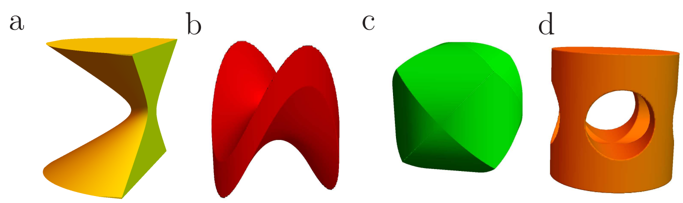
b) (2 points) Match the expressions. There is an exact match.

Integral	Enter A-D
$\int_a^b \int_c^d \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$	
$\int_a^b \vert\vec{r}\,'(t)\vert\ dt$	
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$	
$\int_a^b \int_c^d \vert\vec{r}_u \times \vec{r}_v\vert\ dudv$	

	Type of integral
A	line integral
B	flux integral
C	arc length
D	surface area

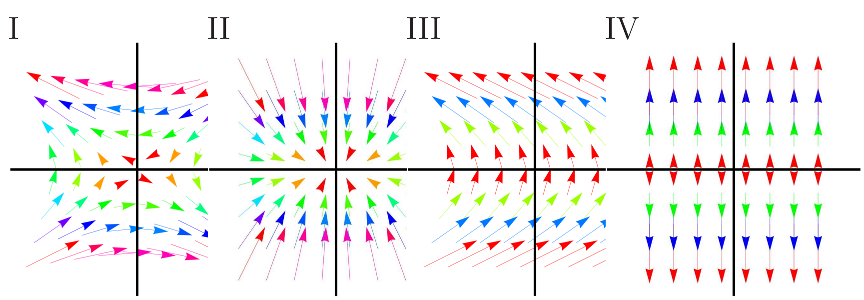
c) (2 points) Match the solids. There is an exact match.

Solid	a-d
$0 \leq x^2 - y^2 - z \leq 1$	
$x^2 + y^2 - z^2 \leq 1, x \geq 0$	
$x^2 + y^2 \leq 3, y^2 + z^2 \geq 1, x^2 + z^2 \geq 1$	
$x^2 + y^2 \leq 1, y^2 + z^2 \leq 1, x^2 + z^2 \leq 1$	



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle 0, 2y\rangle$	
$\vec{F}(x,y) = \langle -x, -2y\rangle$	
$\vec{F}(x,y) = \langle -2y, -x\rangle$	
$\vec{F}(x,y) = \langle -2y, 1\rangle$	



e) (2 points) Match the partial differential equations with formulas and functions $u(t,x)$. There is an exact match.

Equation	1-3	A-C
wave		
heat		
Laplace		

	Formulas
1	$u_{tt} = -u_{xx}$
2	$u_t = u_{xx}$
3	$u_{tt} = u_{xx}$

	Functions
A	$u(t,x) = t + t^2 - x^2$
B	$u(t,x) = t + t^2 + x^2$
C	$u(t,x) = x^2 + 2t$

Solution:
a) 4,2,1,3
b) B,C,A,D
c) b,a,d,c
d) IV,II,I,III
e) 3,2,1 and B,C,A

Problem 3) (10 points)

a) (2 points) Mark the statement or statements which have a zero answer.

Statement	Must be zero
The curl of the gradient $\nabla f(x,y,z)$ at $(1,1,1)$	
The divergence of the curl $\nabla \times \vec{F}(x,y,z)$ at $(1,1,1)$	
The flux of a gradient field $\nabla f(x,y,z)$ through a sphere	
The dot product of $\vec{F}(1,1,1)$ with the curl$(F)(1,1,1)$	
The divergence of a gradient field $\nabla f(x,y,z)$ at $(1,1,1)$	

b) (2 points) Two of the following statements do not make sense. Recall that "incompressible" means zero divergence everywhere and that "irrotational" means zero curl everywhere.

Statement	Makes no sense
The discriminant of the gradient field	
A conservative and incompressible vector field	
The flux of the gradient of a function through a surface	
The gradient of the curl of a vector field	

c) (2 points) Match the following formulas with the geometric object they describe. Fill in the blanks as needed.

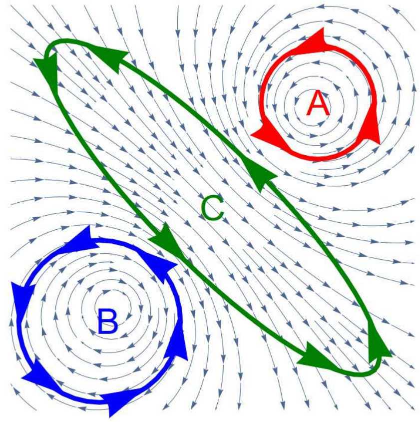
	Geometric object
A	unit tangent vector to a curve
B	unit normal vector to a surface
C	unit normal vector to a level curve
D	curvature of the curve
E	unit tangent vector to the surface

A-E	Formula
	$(\vec{r}_u \times \vec{r}_v)/\lvert\vec{r}_u \times \vec{r}_v\rvert$
	$\nabla f(x,y)/\lvert\nabla f(x,y)\rvert$
	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
	$(\vec{r}_u + \vec{r}_v)/\lvert\vec{r}_u + \vec{r}_v\rvert$
	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

d) (2 points) All three curves in the figure are oriented counter clockwise. Check whether in each of the three cases, the line integral is positive, negative or zero.

Check with a mark which applies. The line integral $\int_\gamma \vec{F} \cdot \vec{dr}$ is

γ	Positive	Negative	Zero
A			
B			
C			



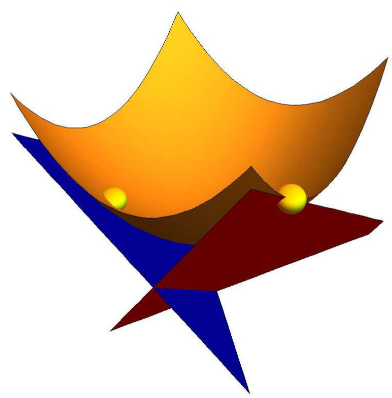
e) (2 points) Line or a plane? That is here the question.

U or V ?	Object
	$\vec{r}(0) + t\vec{B}(\vec{r}(0)) + s\vec{N}(\vec{r}(0))$
	$\vec{r}(0,0) + t\,\vec{r}_u \times \vec{r}_v$

	Dichotomy
U	is a normal plane to a curve
V	is a normal line to a surface

Solution:
a) The first two.
b) The first and last.
c) B,C,A,E,D
d) A - Positive, B - Negative, C - zero e) U,V

Problem 4) (10 points)



a) (6 points) Find the tangent planes at the points $A = (1,1,2)$ and $B = (-2,1,5)$ to the surface

$$x^2 + y^2 - z = 0 \ .$$

b) (4 points) Find the parametric equation of the line of intersection of these two tangent planes.

Solution:
a) $\nabla f = \langle 2x, 2y, -1 \rangle$.
We have $\nabla f(A) = \langle 2, 2, -1 \rangle$ and $\nabla f(B) = \langle -4, 2, -1 \rangle$. The equations of the tangent planes are therefore

$$2x + 2y - z = 2$$

and

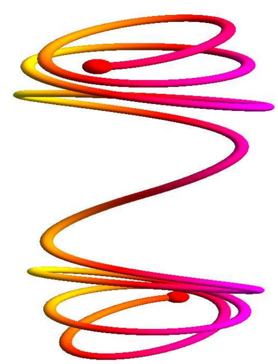
$$-4x + 2y - z = 5 \; .$$

b) In order to find the intersection line, we have to find points P, Q on the intersection and form $\vec{r}(t) = \vec{OP} + t\vec{PQ}$. One can get points by plugging in one variable 0. For example when taking $y = 0$, one gets $P = (-1/2, 0, -3)$ or if $z = 0$, then $Q = (-1/2, 3/2, 0)$. Note that subtracting the two equations gives $x = -1/2$ so that we never can put x=0. Anyway, with the two points just found, we get

$$\vec{r}(t) = \langle -1/2, 0, -3 \rangle + t\langle 0, 1, 2 \rangle \; .$$

Of course there are many different ways to parametrize. A popular answer was also $\vec{r}(t) = \langle -1/2, 3/2 + 6t, 12t \rangle$, as many have computed a vector in the line using the cross product $\langle 2, 2, -1 \rangle \times \langle -4, 2, -1 \rangle = \langle 0, 6, 12 \rangle$, which is an excellent way to get the line too. One still has the task to find a point on the intersection. An other popular intersection point was $(-1/2, 1, -1)$.

Problem 5) (10 points)



Compute the line integral of the vector field

$$\vec{F}(x, y, z) = \langle 2x, 3y^3, 3z^2 \rangle$$

along the curve

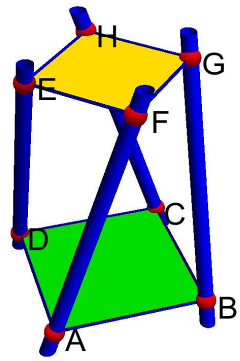
$$\vec{r}(t) = \langle \sin(t)\cos(t^3), \cos(t)\sin(t^3), t \rangle$$

from $t = -\pi/2$ to $t = \pi/2$.

Solution:
This is a typical problem for the fundamental theorem of line integrals. A potential for the vector field is the scalar function $f(x, y, z) = x^2 + 3y^4/4 + z^3$. Now $A = \vec{r}(-\pi/2) = \langle -\cos(\pi^3/8), 0, -\pi/2 \rangle$ and $B = \vec{r}(\pi/2) = \langle \cos(\pi^3/8), 0, \pi/2 \rangle$ are the initial and end point. The result is $f(B) - f(A) = \pi^3/4$.

Problem 6) (10 points)

A monument has the form of a **twisted frustrum**. It is built from a base square platform $\{A = (-10,-10,0), B = (10,-10,0), C = (10,10,0), D = (-10,10,0)\ \}$ connected with an upper square $\{E = (-10,0,20), F = (0,-10,20), G = (10,0,20), H = (0,10,20)\ \}$ using cylinders of radius 1. Find the distance between the pillar going through AF and the pillar going through BG.



Solution:
We have to compute the distance between the two lines and subtract the sum 2 of the two radii. The vector $\vec{v} = \vec{AF} = \langle 10,0,20\rangle$ is parallel to the first pillar. The vector $\vec{w} = \vec{BG} = \angle 0,10,20\rangle$ is parallel to the second pillar. The cross product $\vec{v} \times \vec{w}$ is $\langle -200,-200,100\rangle$. The vector $\vec{AB} = \langle 20,0,0\rangle$ connects one line to the other. The distance between the center lines is given by the volume-area formula $|\vec{AB} \cdot (\vec{v} \times \vec{w})|/|\vec{v} \times \vec{w}|$ which is $40/3$. Now subtract the sum of the radii $1+1$ to get $\boxed{34/3} = 11.3333$.

Problem 7) (10 points)

The monkey saddle gets a cameo: find the volume of the solid contained in the cylinder

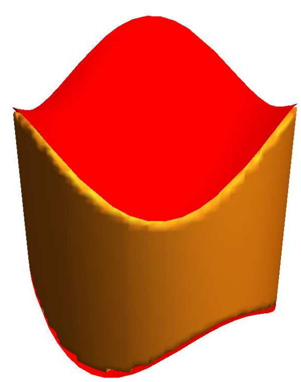
$$x^2 + y^2 < 1$$

above the surface

$$z = x^3 - 3xy^2 - 3$$

and below the surface

$$z = y^3 - 3yx^2 + 3\ .$$



Solution:
Set up the triple integral in cylindrical coordinates:

$$\int_0^{2\pi} \int_0^1 \int_{r^3\cos^3(\theta)-3r^3\cos(\theta)\sin^2(\theta)-3}^{r^3\sin^3(\theta)-3r^3\cos(\theta)\sin^2(\theta)+3} r \; dzdrd\theta$$

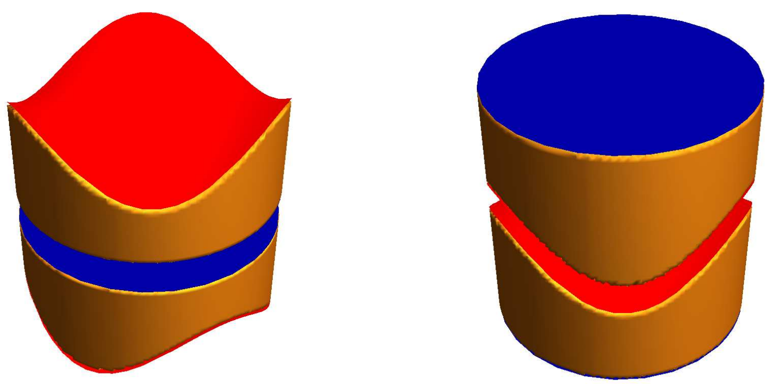
which is

$$\int_0^{2\pi} \int_0^1 6 + r^4(\cos^3(\theta) + 3\cos^2(\theta)\sin(\theta) + 3\cos(\theta)\sin^2(\theta) + \sin^3(\theta)) \; drd\theta \; .$$

One can integrate each term out, or use that each of the occuring trig functions have a symmetry rendering the integrals zero. In any case, we end up with

$$\int_0^{2\pi} \int_0^1 6 \; drd\theta = 6\pi \; .$$

You have seen all integrals which occur in homework: for example $\int \cos^3(t) \; dt = \int \cos(t)(1-\sin^2(t)) \; dt$ or $\int \sin^k(t)\cos(t) \; dt = \sin^{k+1}(t)/(k+1)$ for any integer $k \geq 0$, which if you don't see it directly can be done with the substitution $u = \sin(t)$.
P.S. A few students found an other cool way to compute the volume: cut the solid along the xy plane. Now take the part below and attach it above so that we end up with a cylinder of height 6*. This is how* **Archimedes** *would have done the computation if he would have known about the Monkey saddle.*

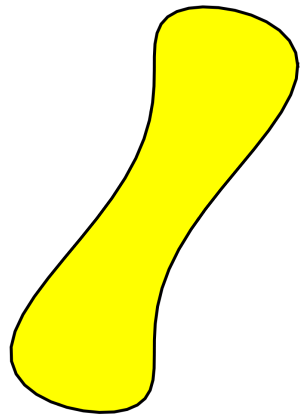


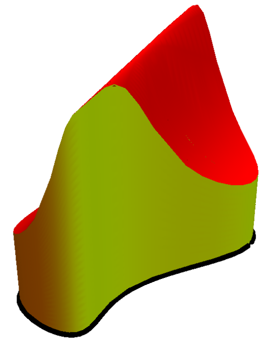
Problem 8) (10 points)

The volume of the solid above the region bound by the planar curve C : $\langle \cos^3(t), \sin(t) + \cos(t) \rangle$ and the graph of $f(x,y) = x^{-2/3}$ is given by the double integral

$$\int\int_G \frac{1}{\sqrt[3]{x^2}} \, dxdy \ .$$

Use the vector field $\vec{F} = \langle 0, 3x^{1/3} \rangle$ to compute this integral.





Solution:
This does at first not look like a Green's theorem problem. A hint was of course that a planar curve and a vector field in the plane was given. Since the curl is the function under consideration, by Greens theorem, we have to compute the line integral of the field $\vec{F}$ along the curve C. It is

$$\int_0^{2\pi} \langle 0, 3\cos(t) \rangle \cdot \langle 3\cos^2(t)\sin(t), \cos(t) - \sin(t) \rangle \, dt$$

which is

$$\int_0^{2\pi} 3\cos^2(t) + 3\cos(t)\sin(t) \, dt = 3\pi \ .$$

The answer is $\boxed{3\pi}$

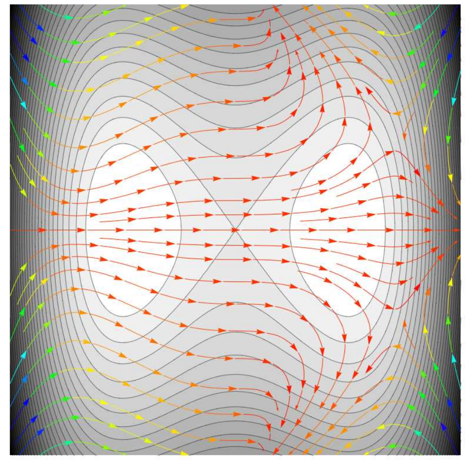
P.S. While we were proud of composing this problem, one discussion point in the team was whether the improper integral would be of any concern. Of course, mathematically there is no problem as $f(x) = x^{-2/3}$ is a function for which the improper integral $\int_{-a}^{b} f(x) \, dx$ exists for every $a < 0 < b$. (See Math 1a or Math 1b) As expected, no student worried about it and we did not expect this also, as the line integral is perfectly smooth. Mathematicians would justify the computation by smoothing out the function f at first (as we did to produce the picture) and then use Green, then take the limit. Integral theorems are a great way to regularize difficult integrals. We have seen in a homework that we can use Green theorem to compute the line integral along a fractal curve like the Koch curve, evenso this curve has infinite length and the line integral does not make sense on the Koch curve, the limit does. An other example is $\vec{F}(x,y) = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$, where the curl is not defined at the origin. It can be seen however as a "generalized function" which is localized at one point only satisfying that any integral containing this point gives one. Green-Stokes-Gauss can be pushed to this theory where it is a theory of "de Rham currents". Using powerful theorems to push computations into domains where it at first does not make sense is a common theme in mathematics and ads to its magic and power.

Problem 9) (10 points)

a) (8 points) The divergence of the vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle 1 - xy^2, 2 + 2yx^2 - yx^4\rangle$$

is the function $f(x,y) = P_x + Q_y$. Find and classify the critical points of $f(x,y)$.



b) (2 points) We have seen in general that the gradient field $\nabla f(x,y)$ is perpendicular to level curves $\{f(x,y) = c\}$ and that $\nabla f(x,y)$ is the zero vector at maxima or minima. Is the vector field $\vec{F}(x,y)$ zero at a maximum of $f(x,y) = \text{div}(F)(x,y)$?

Solution:

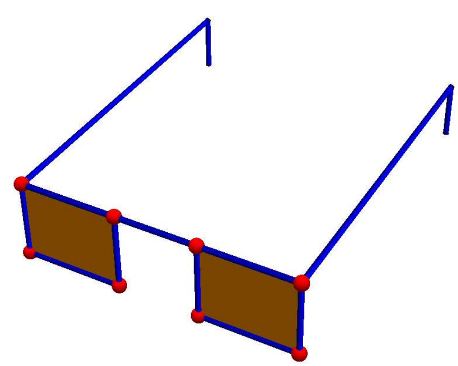
a) The divergence is $f(x,y) = -y^2+2x^2-x^4$. The gradient is $\nabla f(x,y) = \langle 4x(1-x^2), -2y\rangle$. The critical points must satisfy $x = 0, \pm 1$ and $y = 0$. The critical points are $(0,0), (1,0)$ and $(-1,0)$. The second derivative test shows that $(0,0)$ is the saddle point and $(1,0), (-1,0)$ are maxima. We have $f_{xx} = 1 - 12x^2$ and $f_{yy} = -2$ and $f_{xy} = 0$ so that $D = -2(1-12x^2)$ is negative if $x = 0$ and positive $x = \pm 1$.

(x,y)	D	f_{xx}	Type	$f(x,y)$
(-1, 0)	16	-8	maximum	1
(0,0)	-8	4	saddle	0
(1,0)	16	-8	maximum	1

b) Lets just look at the vector field at a maximum like $(1,0)$. It is $\vec{F}(1,0) = \langle 1, 2\rangle$ which is not zero. The answer is $\boxed{\text{No}}$.

Problem 10) (10 points)

Economists know a **constraint duality principle**: "maximizing a first quantity while fixing the second is equivalent to minimizing the second when fixing the first". Let's experiment with that:



a) (5 points) Use the Lagrange method to find the **reading glasses** with maximal glass area $f(x,y) = 2xy$ and fixed frame material $4x + 15y = 120$.

b) (5 points) Use again the Lagrange method to find the reading glasses with minimal frame material $f(x,y) = 4x + 15y$ and fixed glass area $g(x,y) = 2xy = 120$.

Solution:
a) $\nabla f = \langle 2y, 2x\rangle$ and $\nabla g = \langle 4, 15\rangle$. The Lagrange equations are

$$\begin{array}{rcl} 2y & = & 4\lambda \\ 2x & = & 15\lambda \\ 4x + 15y & & 120 \ . \end{array}$$

It leads to the relation $y/x = 4/15$. Plugging in $y = 4x/15$ into the constraints solves the equations and gives $x = 15$ and $y = 4$.
b) We can reuse the same gradients. But now

$$\begin{array}{rcl} 4 & = & 2y\lambda \\ 15 & = & 2x\lambda \\ 2xy & & 120 \ . \end{array}$$

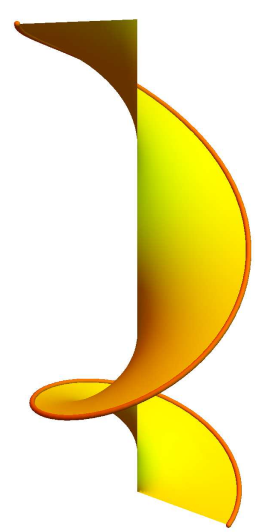
Now, we have again $x = 15$ and $y = 4$.
P.S. Interestingly enough there would be an other solution like $x = -15, y = -4$ which has not appeared in a). But these do not correspond to valid geometries of the glasses. But this example has shown that we can not prove a mathematical duality theorem as **maximizing a first quantity while fixing the second is not totally equivalent to minimizing the second when fixing the first**. A mathematician working in the field of mathematical economics would not get teterred from this experiment and ask "under which conditions does the duality principle hold?" or "is there at all an example, where the duality principle holds?"

Problem 11) (10 points)

a) (4 points) Find the arc length of the **helical curve**

$$\vec{r}(t) = \langle \cos(t), \sin(t), \frac{2t^{3/2}}{3}\rangle \ ,$$

where t goes from 0 to 9.



b) (3 points) Determine the angle between the velocity $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at $t = 0$.

c) (3 points) Write down the surface area integral for the surface $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), 2t^{3/2}/3\rangle$ contained inside the cylinder $x^2 + y^2 \leq 1$ and between $0 \leq z \leq 18$ containing the previous curve in its boundary. You do not have to compute the integral but write down an expression of the form $\int_{\square}^{\square}\int_{\square}^{\square} f(s,t)\, dsdt$ with a function $f(s,t)$ you determine.

Solution:
a) We have $|\vec{r}\,'(t)| = |\langle -\sin(t), \cos(t), \sqrt{t}\rangle| = \sqrt{1+t}$. The arc length is $\int_0^9 \sqrt{1+t}\;dt = (2/3)(10^{3/2}-1)$
b) We have $\vec{r}\;''(t) = \langle -\cos(t), -\sin 9t), t^{-1/2}/2\rangle$. At $t = 0$ we get $\langle\langle 0,1,0\rangle \cdot \langle -1, 0, undefined\rangle$. Actually, one can take the limit $t \to 0$ and get 0. so that the angle is $\pi/2$. We were generous and gave for an "undefined" answer full credit.
c) We have

$$|\vec{r}_u \times \vec{r}_v| = |\langle \sqrt{t}\sin(t), -\sqrt{t}\cos(t), s\rangle| = \sqrt{t+s^2}\;.$$

The integral is

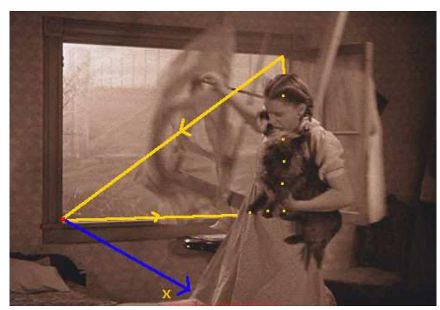
$$\int_0^9 \int_0^1 \sqrt{t+s^2}\;dsdt\;.$$

P.S. By the way, the integral can be done. It would be a good but challenging homework problem. But for an exam problem, with limited time , it would have been bad to ask for evaluating the integral. It would have been a lot of work for 3 points out of 140. The integral $\int \sqrt{1+s^2}ds$ appears quite often, like for example for the arc length of the parabola. You have seen it also in a homework and many of you have worked hard on it to get it. Integrating $\sqrt{t+s^2}$ is similar. Its anti derivative is obtained by integrating by parts and then use a partial fraction trick to get "Marry go round!" the same integral again leading to an equation which can be solved. So, $2\int \sqrt{t+s^2}\;ds = s\sqrt{t+s^2} + t\log(s+\sqrt{t+s^2})$. Now we have to integrate this with respect to t. The first part can be done directly, the second using integration by parts.

Problem 12) (10 points)

The tornado in the **wizard of Oz** induces the **force field**

$$\vec{F} = \langle \cos(x), -2x, y^3 + \sin(z^5)\rangle\;.$$



Dorothy's cardboard is picked up by the storm and pushed along the boundary of the triangle parametrized by

$$\vec{r}(u,v) = \langle 0, u, v\rangle$$

with $0 \le u \le 2$ and $0 \le v \le u/2$. Let C be the boundary of the triangle, oriented counter clockwise when looking from $(1,0,0)$ onto the window. Find the work $\int_C \vec{F} \cdot \vec{dr}$ which the tornado does onto the card board.

Solution:
This is a problem for Stokes theorem. The curl $\vec{G} = \text{curl}(\vec{F})$ is $\langle 2y^2, 0-2 \rangle$. The normal vector to the parametrized surface $\vec{r}(u,v) = \langle 0, u, v \rangle$ is $\langle 1, 0, 0 \rangle$ and points in the right direction. We compute the flux as

$$\int_0^2 \int_0^{y/2} 3y^2 \, dzdy = 6 \; .$$

Problem 13) (10 points)

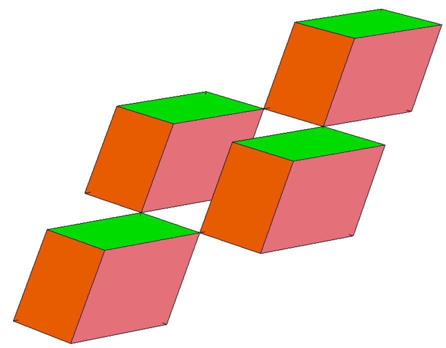
A solid E is the union of 4 congruent, non-intersecting parallelepipeds. One of them is spanned by the three vectors

$$\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \langle 1, 1, 0 \rangle, \vec{w} = \langle 0, 1, 1 \rangle \; .$$

Find the flux of the vector field

$$\vec{F} = \langle 4x + y^{2014}, z^{2014}, x^{2014} \rangle + \text{curl}(\langle -y^{2014}, x^{2014}, 3^{2014} \rangle)$$

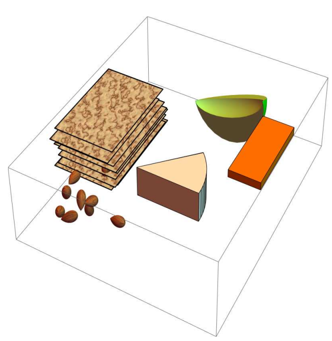
through the outwards oriented boundary surface of E.

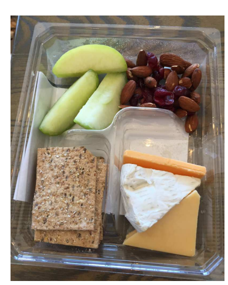


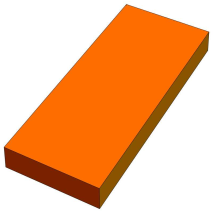
Solution:
The divergence is constant 4. The volume of the parallel epiped is the triple scalar product $\langle 1, 0, 0 \rangle \cdot (\langle 1, 1, 0 \rangle \times \langle 0, 1, 1 \rangle) = 1$. By the divergence theorem, the flux is the triple integral of the divergence over the solid which is 4 times the volume of the solid. Because we have 4 parallel epipeds, the flux is $4 \cdot 4 = 16$.

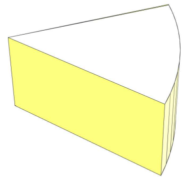
Problem 14) (10 points) No justifications necessary

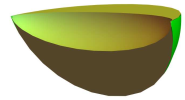
a) (5 points) **Cheese-fruit bistro boxes** can be rarely found in coffee shop these days because a "bistro monster" is eating them all. One of them contains apples, nuts, cheese and crackers. Lets match the objects and volume integrals:



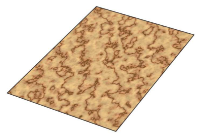



I


II


III

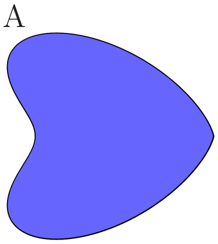

IV


V

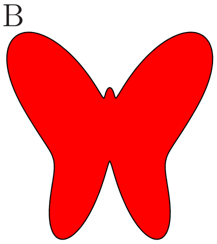
Enter I-V	volume formula
	$\int_1^2 \int_0^{\pi} \int_0^{\pi/5} \rho^2 \sin(\phi)\ d\theta d\phi d\rho$
	$\int_{-6}^{6} \int_{-10}^{10} \int_{-0.1}^{0.1} 1\ dz dx dy$
	$\int_0^{\pi/4} \int_0^{10} \int_{-5}^{5} r\ dz dr d\theta$
	$\int_{-1}^{1} \int_{-4}^{4} \int_{-8}^{8} 1\ dx dy dz$
	$\int_0^{\pi} \int_0^{2\pi} \int_0^{\cos(2\phi)/4} \rho^2 \sin(\phi)\ d\rho d\theta d\phi$

b) (5 points) Biologist **Piet Gielis** once patented polar regions because they can be used to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when matching the following polar regions:

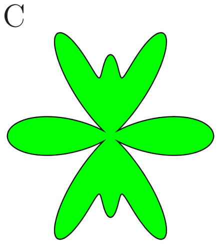
A



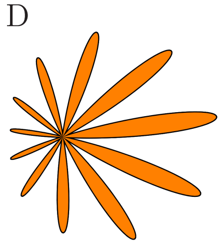
B



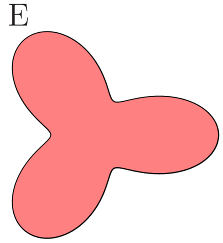
C



D



E



Enter A-E	polar region
	$r(t) \leq \|2 + \cos(3t)\|$
	$r(t) \leq \|\cos(5t) - 5\cos(t)\|$
	$r(t) \leq \|1 + \cos(t)\cos(7t)\|$
	$r(t) \leq \|8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t)\|$
	$r(t) \leq \|\sin(11t) + \cos(t)/2\|$

Solution:
a) III,V,II,I,IV
b) E,A,C,B,D
Symmetry can show the way. If the radial function is even, then there is a symmmetry along the x axes. This happens for A,C,E. Some can be figured out by counting. D has 11 petals. E has 3 petals. The symmetry has decided about B, but C and A were hard to distinguish. One way is to look at $t = \pi$ and $t = 0$.