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TTH 10 Chenglong Yu
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TTH 11:30 Yu-Wen Hsu

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
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8		10
9		10
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11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) T F For any two nonzero vectors $\vec{v}, \vec{w}$ the vector $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v})$ is parallel to $\vec{w}$.

2) T F The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.

3) T F If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t, then the curve is part of a line.

4) T F The curve $\vec{r}(t) = (1-t)A + tB, t \in [0,1]$ connects the point A with the point B.

5) T F For every c, the function $u(x,t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

6) T F The length of the curve $\vec{r}(t) = (t, \sin(t))$, where $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1+\cos^2(t)}\, dt$.

7) T F Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

8) T F The function $f(x,y,z) = x^2 - y^2 - z^2$ decreases in the direction $(2,-2,-2)/\sqrt{8}$ at the point $(1,1,1)$.

9) T F Assume $\vec{F}$ is a vector field satisfying $|\vec{F}(x,y,z)| \leq 1$ everywhere. For every curve $C : \vec{r}(t)$ with $t \in [0,1]$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is less or equal than the arc length of C.

10) T F Let $\vec{F}$ be a vector field which coincides with the unit normal vector $\vec{N}$ for each point on a curve C. Then $\int_C \vec{F} \cdot d\vec{r} = 0$.

11) T F If for two vector fields $\vec{F}$ and $\vec{G}$ one has $\mathrm{curl}(\vec{F}) = \mathrm{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a,b,c)$, where a,b,c are constants.

12) T F If a nonempty quadric surface $g(x,y,z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a,b,c \geq 0$.

13) T F If $\mathrm{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z), then $\mathrm{curl}(\vec{F}) = (0,0,0)$ for all (x,y,z).

14) T F If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

15) T F The divergence of the gradient of any $f(x,y,z)$ is always zero.

16) T F For every vector field $\vec{F}$ the identity $\mathrm{grad}(\mathrm{div}(\vec{F})) = \vec{0}$ holds.

17) T F For every function f, one has $\mathrm{div}(\mathrm{curl}(\mathrm{grad}(f))) = 0$.

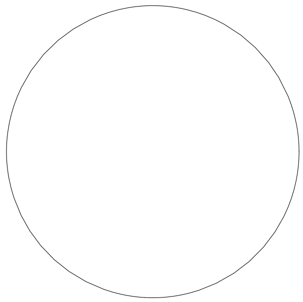
18) T F If $\vec{F}$ is a vector field in space then the flux of $\vec{F}$ through any closed surface S is 0.

19) T F The flux of the vector field $\vec{F}(x,y,z) = (y+z, y, -z)$ through the boundary of a solid region E is equal to the volume of E.

20) T F For every function $f(x,y,z)$, there exists a vector field $\vec{F}$ such that $\mathrm{div}(\vec{F}) = f$.

Problem 2) (10 points)

Problem 2a) (5 points) Match the equations with the objects. No justifications are needed.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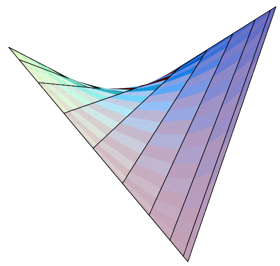
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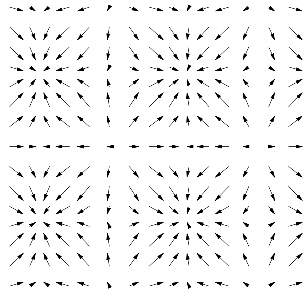
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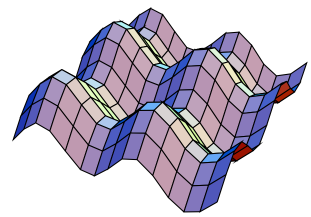
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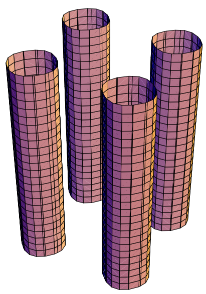
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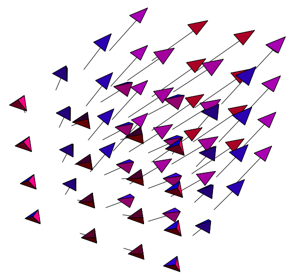
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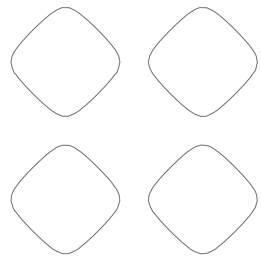
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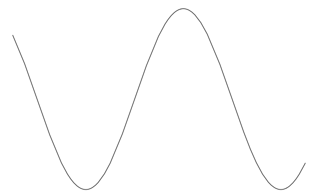
VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x, y, z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$
	$\vec{r}(u, v) = \langle \cos(u), \sin(v), \cos(u)\sin(v) \rangle$
	$\vec{F}(x, y, z) = \langle \cos(x), \sin(x), 1 \rangle$
	$z = f(x, y) = \cos(x) + \sin(y)$
	$g(x, y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x, y) = \langle \cos(x), \sin(y) \rangle$

Problem 2b) (5 points) Mark with a cross in the column below "irrotational" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x, y, z) = (0, 0, 0)$ for all points (x, y, z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z)). No justifications are needed.

Vectorfield	irrotational curl($\vec{F}$) $= \vec{0}$	incompressible div($\vec{F}$) $= 0$
$\vec{F}(x,y,z) = \langle -5, 5, 3 \rangle$		
$\vec{F}(x,y,z) = \langle x, y, z \rangle$		
$\vec{F}(x,y,z) = \langle -y, x, z \rangle$		
$\vec{F}(x,y,z) = \langle x^2+y^2, xyz, x-y+z \rangle$		
$\vec{F}(x,y,z) = \langle x-2yz, y-2zx, z-2xy \rangle$		

Problem 3) (10 points)

a) (2 points) What is the area of the triangle A, B, P, where $A = (1,1,1), B = (1,2,3)$ and $P = (3,2,4)$?

b) (2 points) Find the distance between the point the point P and the line L passing through the points A with B.

Let E be a general parallelogram in three dimensional space defined by two vectors $\vec{u}$ and $\vec{v}$.

c) (3 points) Express the diagonals of the parallelogram as vectors in terms of $\vec{u}$ and $\vec{v}$.

d) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

e) (3 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Problem 4) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x,y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2} .$$

There is a lake in that area as you can see in the photo.

a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.

b) (3 points) For any pair of two different critical points A, B found in a) let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \, d\vec{r}$.

Photo of the lake in the Swiss alps near the Simplon mountain pass.

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8} \, dxdy.$$

Problem 7) (10 points)

In this problem we evaluate $\int \int_D \frac{(x-y)^4}{(x+y)^4} \, dxdy$, where D is the triangular region bounded by the x and y axis and the line $x + y = 1$.

a) (3 points) Find the region R in the uv-plane which is transformed into D by the change of variables $u = x - y, v = x + y$. (It is enough to draw a carefully labeled picture of R.)

b) (3 points) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $(x, y) = (\frac{u+v}{2}, \frac{v-u}{2})$.

c) (4 points) Evaluate $\int \int_D \frac{(x-y)^4}{(x+y)^4} \, dxdy$ using the above defined change of variables.

Hint. The general topic of change of variables does not appear this semester. You can solve the problem nevertheless, when given the formula $\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ for the integration factor (analogous to r when changing to polar coordinates, or $\rho^2 \sin(\phi)$ when going to spherical coordinates). The integral in c) becomes then $\int\int_R u^4/v^4 \; dudv$. The region R is the triangle bounded by the edges $(0,0),(1,1),(-1,1)$.

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function $f(x,y) = -(x^4 - 8x^2 + y^2 + 1)$.
b) (3 points) Classify the critical points.
c) (2 points) Locate the local and absolute maxima of f.
d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

Problem 9) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$.

Problem 10) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = \langle t, \sin t, t^2 \cos t \rangle$ for $0 \leq t \leq \pi$. Let $f(x,y,z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Problem 11) (10 points)

A cylindrical building $x^2 + (y-1)^2 = 1$ is intersected with the paraboloid $z = 4 - x^2 - y^2$.

a) Parametrize the intersection curve and set up an integral for its arc length.

b) Find a parametrization of the surface obtained by intersecting the paraboloid with the solid cylinder $x^2 + (y-1)^2 \leq 1$ and set up an integral for its surface area.

Problem 12) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x, y) = \langle y^2, x^2 \rangle$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0, 0)$, $(1, 0)$ and $(1, 1)$ in two ways:

a) (5 points) by evaluating the three line integrals.
b) (5 points) using Green's theorem.

Problem 13) (10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x, y, z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t), 1 - \cos(t) - \sin(t) \rangle$ with $t \in [0, 2\pi]$.

Problem 14) (10 points)

Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in the xy-plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int\int_S \vec{F} \cdot d\vec{S}$ of the vector field

$$\vec{F} = \langle -4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \rangle$$

through S using the divergence theorem.