9/29/2015 FIRST HOURLY **Math 21a, Fall 2015**

Name:

MWF 9 Jun-Hou Fung
MWF 9 Koji Shimizu
MWF 10 Matt Demers
MWF 10 Dusty Grundmeier
MWF 10 Erick Knight
MWF 11 Oliver Knill
MWF 11 Kate Penner
MWF 12 Yusheng Luo
MWF 12 YongSuk Moon
TTH 10 Will Boney
TTH 10 Peter Smillie
TTH 10 Chenglong Yu
TTH 11:30 Lukas Brantner
TTH 11:30 Yu-Wen Hsu

- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) | T | F | The vector $\langle 2,3,6\rangle$ has a length which is an integer.

Solution:
Yes, its length is equal to 7.

2) | T | F | The surface $x^2+y^2+z^2-2x=3$ is a sphere.

Solution:
Complete the square

3) | T | F | If $\vec{v}\cdot\vec{w}$ is negative, then the angle between $\vec{v}$ and $\vec{w}$ is acute (= smaller than $\pi/2$).

Solution:
True

4) | T | F | The level curves $f(x,y)=1$ and $f(x,y)=0$ do not intersect for $f(x,y)=(xy+\cos(x))^6$.

Solution:
The function would have to be schizophrenic (multi-valued) and have the value 1 and 0 at the same point. This can not happen for a perfectly smooth function like the one given.

5) | T | F | For any nonzero $\vec{a}$, the equation $\vec{a}\times\vec{x}=\vec{b}$ always has a solution $\vec{x}$.

Solution:
We need $\vec{b}$ to be perpendicular.

6) | T | F | For two non-parallel $\vec{a},\vec{b}$, the equation $(\langle x,y,z\rangle\times\vec{a})\cdot\vec{b}=1$ defines a plane.

Solution:
This was not an easy problem. Look at the volume of the parallel epiped spanned by $\vec{a}, \vec{b}$ and $\vec{x}$. This volume is fixed if the vector $\vec{x}$ is on the plane of points which have fixed distance to the plane spanned by $\vec{a}$ and $\vec{b}$.

7) | T | F | The curvature of $\vec{r}(t) = \langle t^3, 1 - t^3, t^3 \rangle$ is 0 if $t = 1$.

Solution:
It is a line

8) | T | F | If $\vec{N}$ is the normal vector and $\vec{T}$ the unit tangent vector to a curve $\vec{r}(t)$ then the vector projection of $\vec{N}(t)$ onto $\vec{T}(t)$ is zero.

Solution:
They are perpendicular.

9) | T | F | There exist non-parallel vectors $\vec{v}, \vec{w}$ such that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

Solution:
Yes, in that case, one of the vectors has to be zero.

10) | T | F | The point given in spherical coordinates as $\rho = 3, \phi = \pi/2, \theta = \pi$ is on the x-axes.

Solution:
It is the north pole.

11) | T | F | The parametrized curve $\vec{r}(t) = \langle 5\cos(3t), 3\sin(3t), 0 \rangle$ is an ellipse.

Solution:
Indeed, and it is contained in the xz-plane.

12) | T | F | If the vector projection of $\vec{v}$ onto $\vec{w}$ is $\vec{w}$ then $\vec{v} = \vec{w}$.

Solution:
We can take $\vec{v}$ plus an orthogonal vector.

13) | T | F | Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \times \vec{v}) \times \vec{w}| \leq |\vec{u}||\vec{v}||\vec{w}|$.

Solution:
Use the identity for the length of the cross product

14) | T | F | The surface $y^2 + z = x^2$ is a hyperbolic paraboloid.

Solution:
Look at the traces

15) | T | F | The curvature of a curve $\vec{r}(t)$ at time $t = 0$ is the same as the curvature of $\vec{r}(\sin(t))$ at time $t = 0$.

Solution:
This is a re-parametrization

16) | T | F | The arc length of the curve $\langle \sin(t), 0, \cos(t) \rangle$ from $t = 0$ to $t = 2\pi$ is equal to 2π.

Solution:
It is a circle

17) | T | F | The curve $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(t) + \sin(t) \rangle$ is on the intersection of $x^2 + y^2 = 1$ and $x + y - z = 0$.

Solution:
Just plug in

18) | T | F | Using $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle$, the identity $(\vec{i} \times \vec{j}) \times \vec{j} = \vec{i} \times (\vec{j} \times \vec{j})$ holds.

Solution:
Indeed, associativity fails.

19) | T | **F** |

Using the same notation, the identity $(\vec{i} \cdot \vec{j})\vec{j} = \vec{i}(\vec{j} \cdot \vec{j})$ holds.

Solution:
Indeed, associativity fails.

20) | T | **F** |

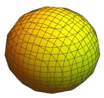
$\langle \cos t, \sin t, t \rangle$, $0 \leq t \leq 2$ and $\langle \cos(t^3), \sin(t^3), t^3 \rangle$, $0 \leq t \leq 2$ have the same arc length.

Solution:
While it is true that arc length is invariant under reparametrization, we have in this case a different curve because the end points do not agree. If the time interval would have been $[0, 1]$ (like in some practice exam), then the arc length would agree.

| | Total |

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contours $g(x, y, z) = 1$. Enter O, if there is no match.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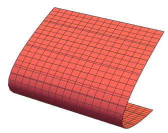
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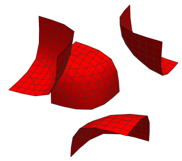
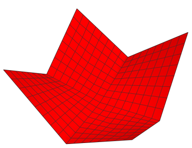

III

Function $g(x, y, z) = 1$	Enter O,I,II or III
xyz	
$x^2 + y^2 + z^2$	
$z^2 - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.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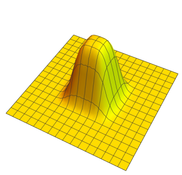
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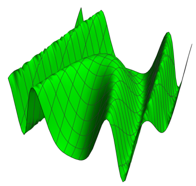
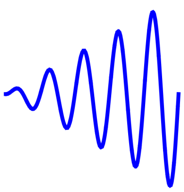

III

Function $f(x, y) =$	Enter O,I,II or III
$\cos(x^2 + y)$	
$\lvert x + y\rvert + \lvert xy\rvert$	
$\exp(-x^4 - y^4)$	
x^3	

c) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.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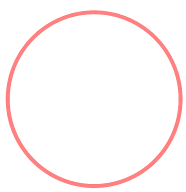
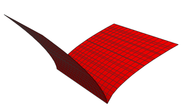

II


III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\vec{r}(t) = \langle t, t\sin(5t)\rangle$	
$\vec{r}(t) = \langle t\sin(5t), t\rangle$	
$\vec{r}(t) = \langle \sin(5t), \cos(5t)\rangle$	
$\vec{r}(t) = \langle \cos(5t), \cos(5t)\rangle$	

d) (2 points) Match functions g with level surface $g(x, y, z) = 1$. Enter O, if there is no match.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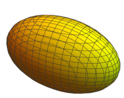
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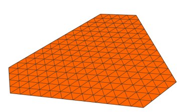
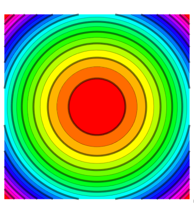

III

Function $g(x, y, z) = 1$	Enter O, I,II or III
$x^2 - y^2 + z^2 = 1$	
$x - y - z = 1$	
$y^3 = z^2$	
$x^2/4 + y^2 + z^2/2 = 1$	

e) (2 points) Match the contour maps to a function $f(x, y)$. Enter O if no match.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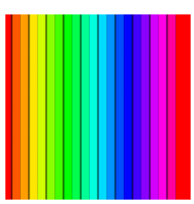
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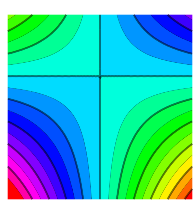
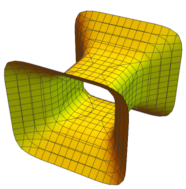

III

$f(x, y) =$	Enter O,I,II or III
$x^2 - y^4$	
$xy - x$	
x	
y	
$x^2 + y^2$	

Solution:
a) III,I,II,0
b) III,I,II,0
c) I,III,II,0
d) 0,III,I,II
e) 0,III,II,0,I

Problem 3) (10 points)	(Only answers are needed)

a) (4 points) The following contour surfaces were deformed by setting $X = x^3, Y = y^3, Z = z^3$. Can you label the original quadrics from which it was deformed?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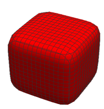
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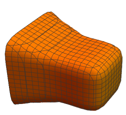
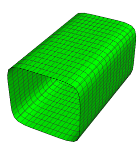

III


IV

Surface	I-IV	name A-D
$X^2 + Y^2 + Z^2 = 1$		
$X + Y^2 + Z^2 = 1$		
$X^2 - Y^2 + Z^2 = 1$		
$X^2 + Z^2 = 1$		

Fill in:
A) Ellipsoid'esque
B) Paraboloid'esque
C) Hyperboloid'esque
D) Cylinder'esque

b) Help Larry the physicist in the movie "A serious man", to compute some quantities:

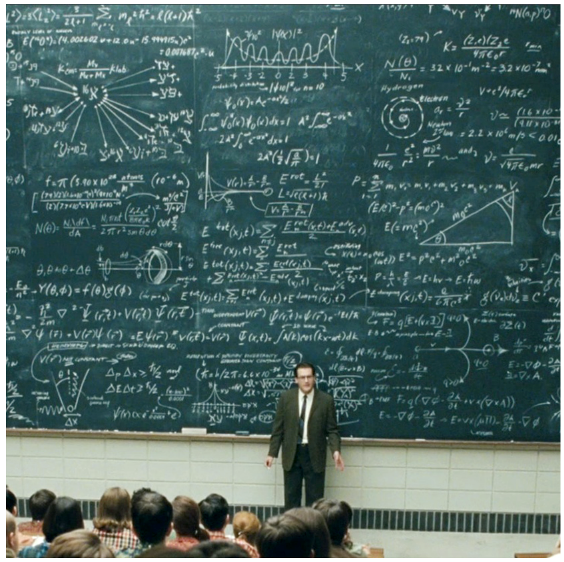
$\vec{v} = \langle 1, 2, 3 \rangle$ represent the velocity
$\vec{w} = \langle 0, 1, 1 \rangle$ represent angular velocity
$\vec{B} = \langle 1, 0, 1 \rangle$ represent magnetic field
$\vec{r} = \langle 0, 0, 1 \rangle$ represent position. Compute:

(i) (2 points) Coriolis force $\vec{v} \times \vec{w}$.

(ii) (2 points) Lorentz force $\vec{v} \times \vec{B}$.

(iii) (1 point) Kinetic energy $(\vec{v} \cdot \vec{v})/2$.

(iv) (1 point) Magnetic energy $\vec{B} \cdot \vec{B}/2$.



Larry: "I mean - even I don't understand the dead cat. The math is how it really works."

(i)

ii)

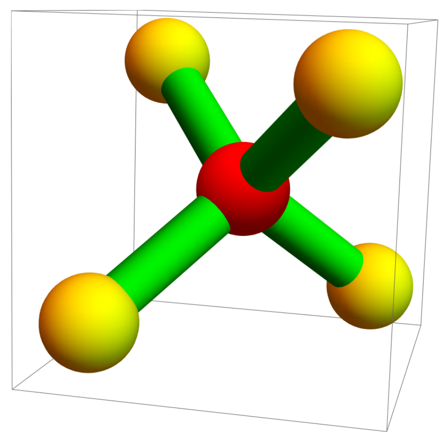
(iii)

iv)

Solution:
a) II A
III,B
I,C
IV,D
b) $< -1, -1, 1 >$
$< 2, 2, -2 >$
7
1

Problem 4) (10 points)

Methane CH_4 is the number two greenhouse gas emitted by human activity in the US. The four hydrogen atoms of **methane** are located at the vertices $P = (2,2,2), Q = (2,0,0), R = (0,2,0), S = (0,0,2)$ and form a regular tetrahedron, while C is the central carbon atom located at $(1,1,1)$.



a) (2 points) Find one bond distance $|CP|$ and the distance $|PQ|$

b) (4 points) Find the cosine of the bond angle between $\overrightarrow{PC}$ and $\overrightarrow{PQ}$.

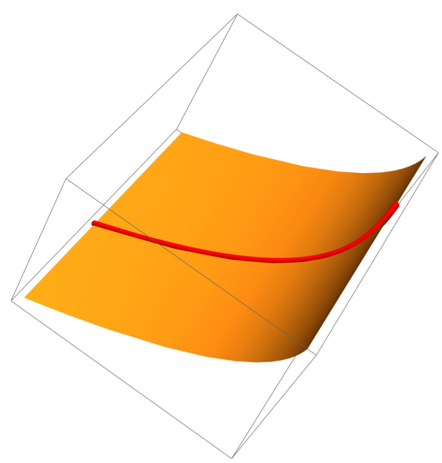
c) (4 points) What is volume of the parallelepiped spanned by $\overrightarrow{PC}, \overrightarrow{QC}, \overrightarrow{RC}$?

Solution:
a) $|CP| = \sqrt{3}$
$|PQ| = 2\sqrt{2}$
b) We use the basic cos-formula for getting the angle. $\cos(\theta) = \sqrt{6}/3 = \sqrt{2}/\sqrt{3}$.
c) We compue the triple scalar product between the three vectors. The answer is 4. It makes sense since it is just half the volume of the cube of side length 8.

Problem 5) (10 points)

Consider the curve

$$\vec{r}(t) = \langle 2e^t, t, e^{2t} \rangle \ .$$



a) (3 points) Compute the speed $|\vec{r}\,'(0)|$.

b) (5 points) Find the arc length from $t = -2$ to $t = 1$.

c) (2 points) There exists a constant a such that the curve lies on the cylindrical paraboloid $x^2 = az$. Which a does apply?

Solution:
a) The velocity is $\vec{r}\,'(t) = \langle 2e^t, t, e^{2t} \rangle$. At $t = 0$, this is $\vec{r}\,'(0) = \langle 2, 0, 1 \rangle$.
b) We have to integrate

$$\int_{-2}^{1} \sqrt{4e^{2t} + 1 + 4e^{4t}}\; dt$$

The term inside the square root can be written as $(2e^{2t} + 1)^2$. The integration is now elementary and becomes $e^2 - e^{-4} + 3$
c) Since $x = 2e^t$ and $z = e^{2t}$ we see $x^2 = 4e^{2t} = az$ for $a = 4$.

Problem 6) (10 points)

The highest **bungee jump** ever recorded was done from the 233 meter high Macau Tower. Assume the rope pulls back with a force $2t$ so that the acceleration is

$$\vec{r}\,''(t) = \langle 0, 0, 2t - 10 \rangle \ .$$

Assume the initial velocity is $\langle 1, 0, 0 \rangle$ and that the daredevil jumps from $\vec{r}(0) = \langle 0, 0, 233 \rangle$:

a) (5 points) Find $\vec{r}\,'(t)$ and determine t_0 for which the third component $z'(t_0) = 0$. This is the time of the lowest point.

b) (5 points) Find $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $\vec{r}(t_0)$. Did the jumper hit the ground $z = 0$?

Solution:
a) $\vec{r}' = \langle 0, 0, t^2 - 10t \rangle + \langle C_1, C_2, C_3 \rangle$.
Fixing the initial velocity shows $\vec{r}' = \langle 1, 0, t^2 - 10t \rangle$. The roots of $t^2 - 10t$ are $t = 0$ and $t = 10$.The first one is the start of the jump, the second one the end.
b) Integrate again and fix the constants. We get $\vec{r} = \langle t, t^3/3 - 5t - 233 \rangle$
At time $t = 10$, we are at $\vec{r}(10) = \langle 10, 0, 199/3 \rangle$. The jumper survives. The situation modeled here is pretty close to what you see in the video. The jumpers don't get too close to the ground. The t part of the acceleration assumes that the rope satisfies the Hook law telling that the force pulling back gets bigger if is is getting longer. After that moment, the differential equation does not model the situation any more well.

Problem 7) (10 points)

The **logarithmic spiral** is parametrized by $\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), 0 \rangle$.

a) (5 points) Find the angle between $\vec{r}\,'(t)$ and acceleration $\vec{r}\,''(t)$ at time $t = 0$.

b) (5 points) Compute the curvature at $t = 0$.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} .$$

One miracle about the spiral is that arc length from 0 to t multiplied with curvature at t is constant. Jacob Bernoulli called it the curve the "Spira mirabilis" which means "miraculous spiral".

Solution:

a) $\vec{r}\,'(t) = \langle e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t)), 0 \rangle$ $\vec{r}\,''(t) = \langle 2e^t \sin(t), 2e^t \cos(t), 0 \rangle$

$\vec{r}\,'(0) = \langle 1, 1, 0 \rangle$

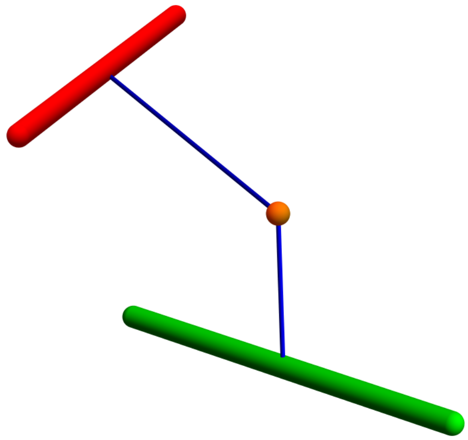
$\vec{r}\,''(0) = \langle 0, 2, 0 \rangle$.

$\theta = \arccos(1/\sqrt{2}) = \pi/4$

b) We compute $\langle 1, 1, 0 \rangle \times \langle 0, 2, 0 \rangle = \langle 0, 0, 2 \rangle$. Therefore, $\kappa = |\langle 1, 1, 0 \rangle \times \langle 0, 2, 0 \rangle|/2 = \sqrt{2}/2 = 1/\sqrt{2}$

Problem 8) (10 points)

Given a line $\vec{r}(t) = \langle 1 + t, t, t \rangle$ and a line $\vec{s}(t) = \langle 1 - t, 1 + t, 1 - t \rangle$.



a) (6 points) Find the sum of the distances of the point $(0, 0, 0)$ to the two lines.

b) (4 points) Find the distance between the two lines.

Solution:
This is a routine distance problem, but three problems.
a) The first distance is $|\langle 1,0,0\rangle \times \langle 1,1,1\rangle|/\sqrt{3} = \sqrt{2}/\sqrt{3}$. The second distance is $|\langle 1,1,1\rangle \times \langle -1,1,-1\rangle|/\sqrt{3} = \sqrt{8}/\sqrt{3}\ \sqrt{6}$. In both cases, we have computed the area of a parallelogram and divided by the base length.
b) Also here, we think about geometry and get the volume of the parallel epiped spanned by $\langle 1,1,1\rangle, \langle -1,1,-1\rangle$ and $\langle 0,1,1\rangle$ (which is 2) divided by the area of the parallelogram spanned by $\langle 1,1,1\rangle, \langle -1,1,-1\rangle$ which is the length of their cross product (which is $\sqrt{4+4} = \sqrt{8}$. The answer is $2/\sqrt{8} = 1/\sqrt{2}$

Problem 9) (10 points)

Parametrize the following surfaces in space. As usual r, θ, z are cylindrical and ρ, θ, ϕ are spherical coordinate variables. You do not need to give bounds on the parameters.

a) (2 points) Parametrize $y = \cos(3x) - \sin(3z)$ as

$\vec{r}(x, z) =$

b) (2 points) Parametrize $\rho = 2 + \cos(8\theta + 5\phi)$ as

$\vec{r}(\theta, \phi) =$

c) (2 points) Parametrize $r^2 - z^2 = 1$ as

$\vec{r}(\theta, z) =$

d) (2 points) Parametrize $x = 0$ as

$\vec{r}(y, z) =$

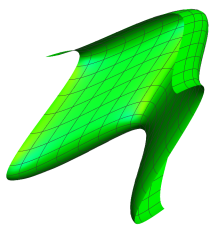
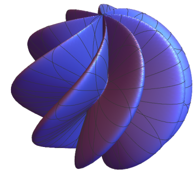
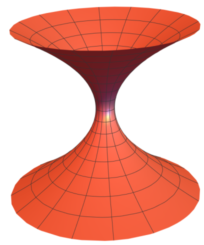
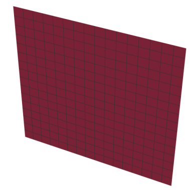
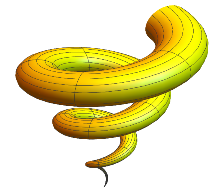
e) Decide whether none, one, or both of the grid curves u = 1, v = 1 is a circle, if

$$\vec{r}(u, v) = \langle (3u + u\cos(v))\cos(2u), (3u + u\cos(v))\sin(2u), (3u + u\sin(v))\rangle$$

(1 point) Is the curve $u = 1$ a circle? | Yes or No

(1 point) Is the curve $v = 1$ a circle? | Yes or No

Illustrations:

a) b) c) d) e)

Solution:

a) $\langle x, \cos(3x) - \sin(3z), z\rangle$.

b) $\langle (2+\cos(8\theta+5\phi))\sin(\phi)\cos(\theta), (2+\cos(8\theta+5\phi))\sin(\phi)\sin(\theta), (2+\cos(8\theta+5\phi))\cos(\phi)\rangle$.

c) $\sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z\rangle$.

d) $\langle 0, y, z\rangle$ e) First "Yes" then "No". For seeing that $u = 1$ is a circle, you can write down $\vec{r}(1, v)$ and see that $x^2 + y^2 + z^2 = 1$ and $x\sin(1) - y\cos(1) = 0$ which means that the curve is on the intersection of a sphere with a plane. The curve $v = 1$ is clearly a spiral lying on the cone $x^2 + y^2 = z^2$. (To see that the first one was a circle was probably the hardest point to earn in this exam!)