9/29/2015 FIRST HOURLY PRACTICE IX **Math 21a, Fall 2015**

Name:

MWF 9 Jun-Hou Fung
MWF 9 Koji Shimizu
MWF 10 Matt Demers
MWF 10 Dusty Grundmeier
MWF 10 Erick Knight
MWF 11 Oliver Knill
MWF 11 Kate Penner
MWF 12 Yusheng Luo
MWF 12 YongSuk Moon
MWF 12 Will Boney
TTH 10 Peter Smillie
TTH 10 Chenglong Yu
TTH 11:30 Lukas Brantner
TTH 11:30 Yu-Wen Hsu

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | The point $(x, y, z) = (-1, -1, -1)$ is in spherical coordinate described as $(\rho, \theta, \phi) = (\sqrt{3}, 5\pi, 3\pi/4)$

Solution:
Make a picture and look at the angles. The θ angle is false.

2) | T | **F** | If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$.

Solution:
No, the vectors can be parallel without beeing zero.

3) | T | **F** | The surface $z^2 + 4y^2 = x^2 + 1$ is a two sheeted hyperboloid.

Solution:
It is a deformed one-sheeted hyperboloid.

4) | **T** | F | The surface $4x^2 - 4x + y^2 - 2y - 120 = -z^2$ is an ellipsoid.

Solution:
Complete the square

5) | **T** | F | The parametrized lines $\vec{u}(t) = \langle 1 + 2t, 2 - 5t, 1 + t\rangle$ and $\vec{v}(t) = \langle 3 - 4t, -3 + 10t, 2 - 2t\rangle$ are the same line.

Solution:
The vectors are parllel, and both lines go through the same point.

6) | **T** | F | The surface $\sin(x) = z$ contains lines which are parallel to the y-axis.

Solution:
One can translate the surface in the y direction.

7) T **F** If $\vec{u} \cdot \vec{v} = 0$, $\vec{v} \cdot \vec{w} = 0$ and $\vec{v}$ is not the zero vector, then $\vec{u} \cdot \vec{w} = 0$.

Solution:
The assumption means that $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{w}$. But that does not mean that $\vec{u}$ and $\vec{w}$ are perpendicular.

8) T **F** The curvature of a curve depends upon the speed at which one travels upon it.

Solution:
The curvature does not depend on the parametrization.

9) T **F** Two lines in space that do not intersect must be parallel.

Solution:
They can be skew.

10) T **F** A line in space can intersect an elliptic paraboloid in 4 points.

Solution:
It can only intersect it in 2 points or 1 point or avoid it at all.

11) **T** F If $\vec{u} \times \vec{v} = 0$ and $\vec{u} \cdot \vec{v} = 0$, then one of the vectors $\vec{u}$ and $\vec{v}$ is zero.

Solution:
A vector which is both parallel and perpendicular to an other vector can only be the zero vector.

12) **T** F If the velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are parallel at time $t = 1$, then the curvature $\kappa(t)$ of the curve is zero at time $t = 1$.

Solution:
You can see this from the formula $\kappa = |r'(t) \times r''(t)|/|r'(t)|^3$. You can also think about it as follows. Assume the curvature were $\kappa = 1/r$. Then you as well locally move on a circle with radius r. But the acceleration has now a component perpendicular to your velocity vector. But we assumed there is no such acceleration.

13) | T | **F** | If the speed of a parametrized curve is constant over time, then the curvature of the curve $\vec{r}(t)$ is zero.

Solution:
It would be true if the velocity would be constant over time. But we can move on a circle with constant speed.

14) | T | **F** | The length of the vector projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is always equal to the length of the vector projection of $\vec{w}$ onto $\vec{v}$.

Solution:
If the lengths of $\vec{v}$ and $\vec{w}$ are the same, then the statement is true. In general, it is not.

15) | **T** | F | A quadric $ax^2 + by^2 + cz^2 = 1$ is contained in the interior of a sphere $x^2 + y^2 + z^2 < 100$, then the constants a, b, c are all positive and the quadric is an ellipsoid.

Solution:
If any of the constants would become negative, the quadric becomes unbounded.

16) | T | **F** | There is a hyperboloid of the form $ax^2 + by^2 - cz^2 = 1$ which has a trace which is a parabola.

Solution:
Traces are either hyperbola or ellipses.

17) | **T** | F | The set of points in space which have distance 1 from the line $x = y = z$ form a cylinder.

Solution:
Yes, if the the line is the z-axis, then $x^2 + y^2 = 1$ is the equation of the cylinder.

18) T F The velocity vector of a parametric curve $\vec{r}(t)$ always has constant length.

Solution:
This is only true for an arc length parametrization.

19) T F The volume of a parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is $|(\vec{u} \times \vec{v}) \times \vec{w}|$.

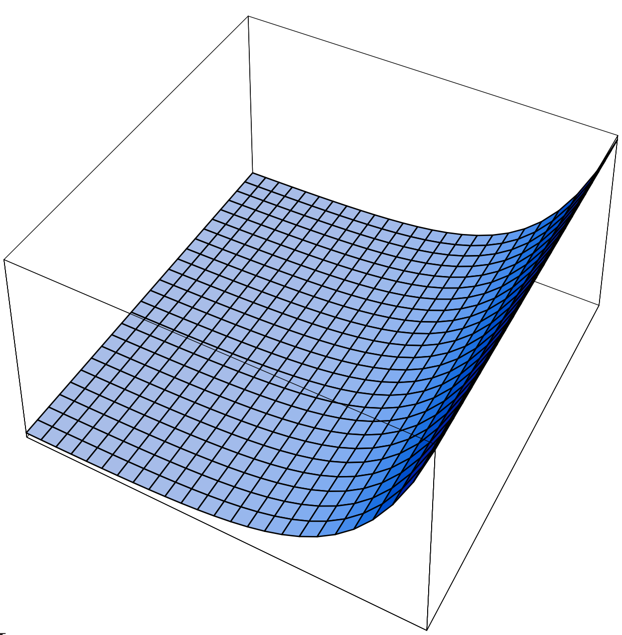
Solution:
The triple scalar product contains also a dot product.

20) T F The equation $x^2 + y^2/4 = 1$ in space describes an ellipsoid.

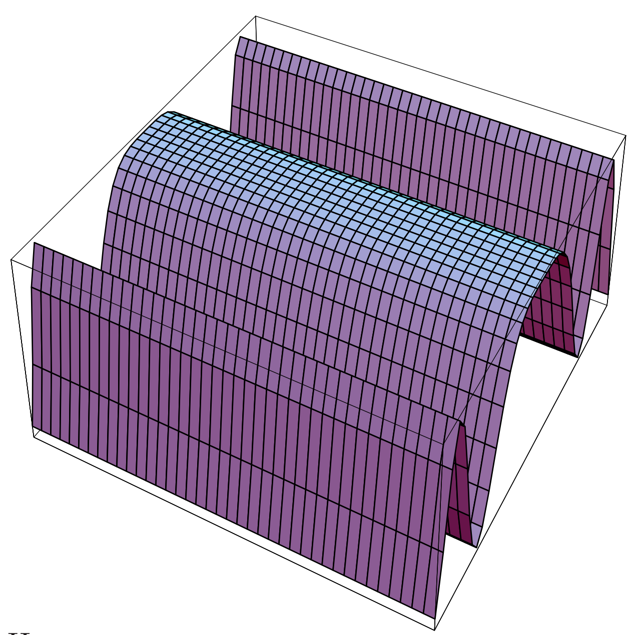
Solution:
The equation describes an elliptical cylinder.

Problem 2a) (3 points)

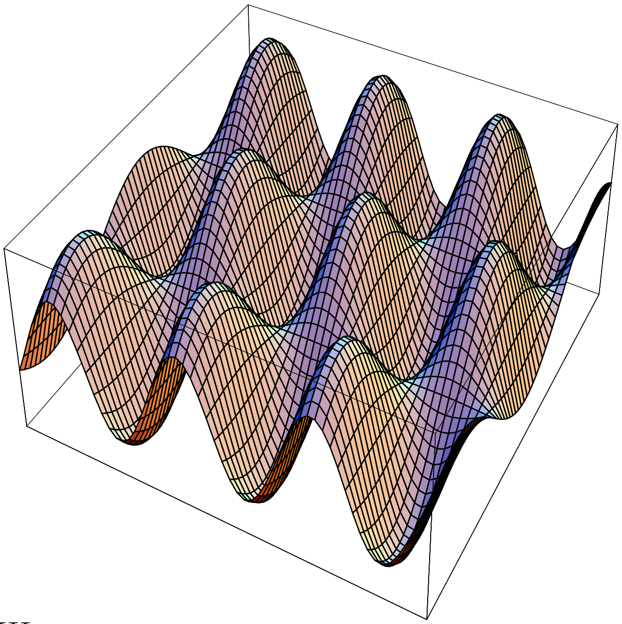
Match the equation with their graphs. No justifications are needed.



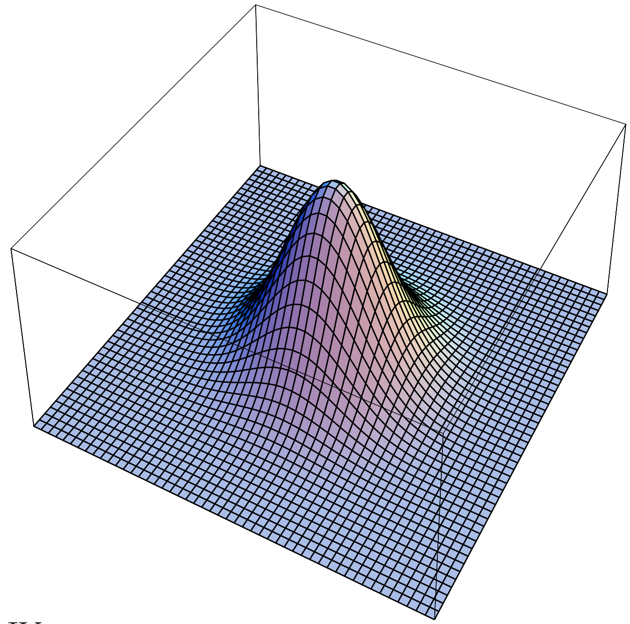
I



II



III



IV

Enter I,II,III,IV here	Equation
	$z = \sin(5x)\cos(2y)$
	$z = \cos(y^2)$
	$z = e^{-x^2-y^2}$
	$z = e^x$

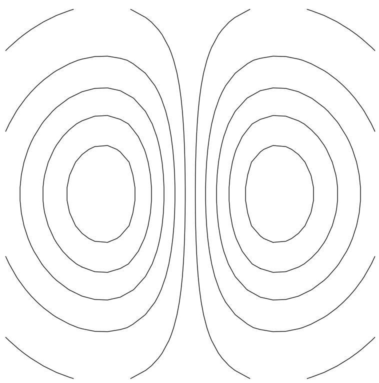
Solution:

Enter I,II,III,IV here	Equation	Justification
III	$z = \sin(5x)\cos(2y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
IV	$z = e^{-x^2-y^2}$	has a maximum at (0,0)
I	$z = e^x$	no y dependence, monotone in x

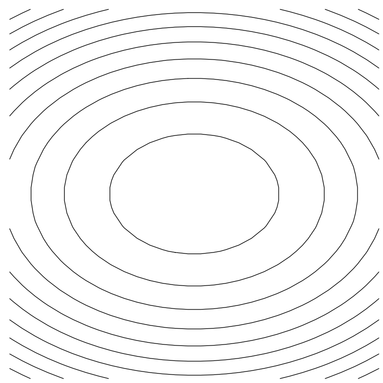
Problem 2b) (4 points)

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. No justifications are needed.

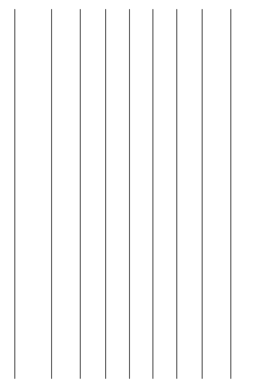
I



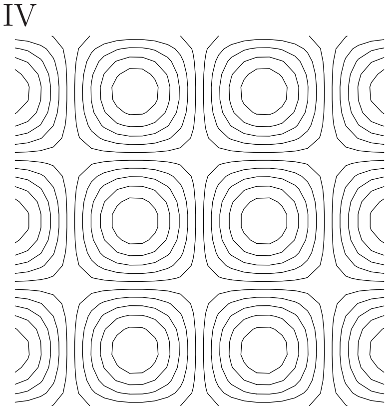
II



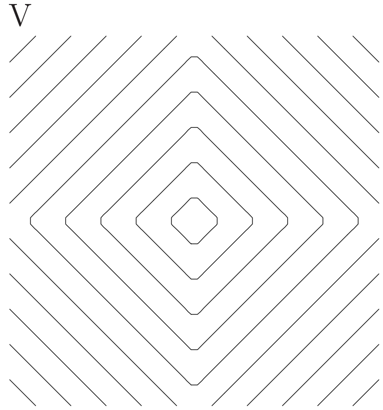
III



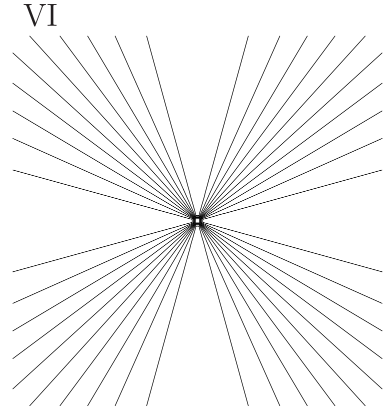
IV



V



VI



Enter I,II,III,IV,V or VI here	Function $f(x, y)$
	$f(x, y) = \sin(x)$
	$f(x, y) = x^2 + 2y^2$
	$f(x, y) = \lvert x\rvert + \lvert y\rvert$
	$f(x, y) = \sin(x)\cos(y)$
	$f(x, y) = xe^{-x^2-y^2}$
	$f(x, y) = x^2/(x^2 + y^2)$

Solution:

Enter I,II,III,IV,V or VI here	Function $f(x,y)$
III	$f(x,y) = \sin(x)$
II	$f(x,y) = x^2 + 2y^2$
V	$f(x,y) = \|x\| + \|y\|$
IV	$f(x,y) = \sin(x)\cos(y)$
I	$f(x,y) = xe^{-x^2-y^2}$
VI	$f(x,y) = x^2/(x^2+y^2)$

Problem 2c) (3 points)

Match the following points in cartesian coordinates with the points in spherical coordinates:

a) $(x, y, z) = (\sqrt{2}, 0, 0)$
b) $(x, y, z) = (0, \sqrt{2}, 0)$
c) $(x, y, z) = (0, 0, \sqrt{2})$
d) $(x, y, z) = (1, 1, 0)$
e) $(x, y, z) = (1, 0, 1)$
f) $(x, y, z) = (0, 1, 1)$

1) $(\rho, \phi, \theta) = (\sqrt{2}, 0, 0)$.
2) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/4)$.
3) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, 0)$.
4) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/2, \pi/2)$.
5) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, \pi/2)$.
6) $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, 0)$.

Solution:
a = 3, b = 4, c = 1, d = 2, e = 6, f = 5

Problem 3) (10 points)

a) (7 points) Find a parametric equation for the line which is the intersection of the two planes $2x - y + 3z = 9$ and $x + 2y + 3z = -7$.

b) (3 points) Find a plane perpendicular to both planes given in a) which has the additional property that it passes through the point $P = (1, 1, 1)$.

Solution:
a) We get the direction of the line by taking the cross product of $\langle 2, -1, 3\rangle$ and $\langle 1, 2, 3\rangle$ which is $\langle -9, -3, 5\rangle$. To find a point in both lines, subtract one from the other to get $x - 3y = 16$. If $z = 0$, then $2x - y = 9$ and $x + 2y = -7$ so that $x = 11/5, y = -23/5$. The parametric equations are $(x, y, z) = (11/5, -23/5, 0) + t(-9, -3, 5)$.
b) Plug in the coordinates $(x, y, z) = (1, 1, 1)$ of the point to get the constant $-9x - 3y + 5z = -7$.

Problem 4) (10 points)

Given the vectors $\vec{v} = \langle 1, 1, 0\rangle$ and $\vec{w} = \langle 0, 0, 1\rangle$ and the point $P = (2, 4, -2)$. Let Σ be the plane which goes through the origin $(0, 0, 0)$ and which contains the vectors $\vec{v}$ and $\vec{w}$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

a) (6 points) Compute the distance from P to the plane Σ.

b) (4 points) Find the shortest distance from P to the sphere S.

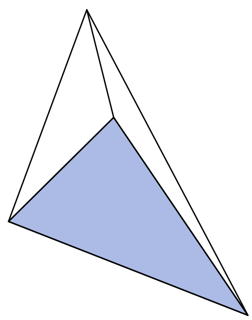
Solution:
a) $\Sigma : x - y = 0$, $n = (1, -1, 0)$. The point $Q = (0, 0, 0)$ is on the plane. $\vec{PQ} \cdot \vec{n}/|\vec{n}| = \langle 2, 4, -2\rangle \cdot \langle 1, -1, 0\rangle/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$ is the distance.
b) $\sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$ is the distance to the origin. So the distance to the sphere is 1 less. The answer is $\sqrt{24} - 1$.

Problem 5) (10 points)

a) (6 points) Find an equation for the plane through the points $A = (0, 1, 0), B = (1, 2, 1)$ and $C = (2, 4, 5)$.

b) (4 points) Given an additional point $P = (-1, 2, 3)$, what is the volume of the tetrahedron which has A, B, C, P among its vertices.

A useful fact which you can use without justification in b): the volume of the tetrahedron is 1/6 of the volume of the parallelepiped which has AB, AC, and AP among its edges.



Solution:
a) The vectors $\vec{v} = \vec{AB} = \langle 1, 1, 1 \rangle$ and $\vec{w} = \vec{AC} = \langle 2, 3, 5 \rangle$ are in the plane. Their cross product is $\vec{n} = \langle 2, -3, 1 \rangle$. This vector is perpendicular to the plane. The equation of the plane is therefore $2x - 3y + z = d$. Plugging in one point like A, gives $d = -3$.
b) With the vector $\vec{u} = \vec{AP} = \langle -1, 1, 3 \rangle$, one can express the volume of the parallelepiped as $|[\vec{u}, \vec{v}, \vec{w}]| = |\vec{u} \cdot \vec{n}| = |\langle -1, 1, 3 \rangle \cdot \langle 2, -3, 1 \rangle| = |2| = 2$. The volume of the tetrahedron is $2/6 = 1/3$.

Problem 6) (10 points)

The parametrized curve $\vec{u}(t) =< t, t^2, t^3 >$ (known as the "twisted cubic") intersects the parametrized line $\vec{v}(s) = \langle 1 + 3s, 1 - s, 1 + 2s \rangle$ at a point P. Find the angle of intersection.

Solution:
The curves intersect at $P = (1, 1, 1)$ with $t = 1$, $s = 0$ So, it remains to find the angle between the velocity vectors $\vec{u}\,'(= \langle 1, 2, 3 \rangle$ and $\vec{v}\,'(0) = \langle 3, -1, 2 \rangle$, which is 60 degrees.

Problem 7) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (\log(t), 2t, t^2)$, where $\log(t)$ is the natural logarithm (denoted by $\ln(t)$ in some textbooks).

a) What is the velocity and what is the acceleration at time $t = 1$?

b) Find the length of the curve from $t = 1$ to $t = 2$.

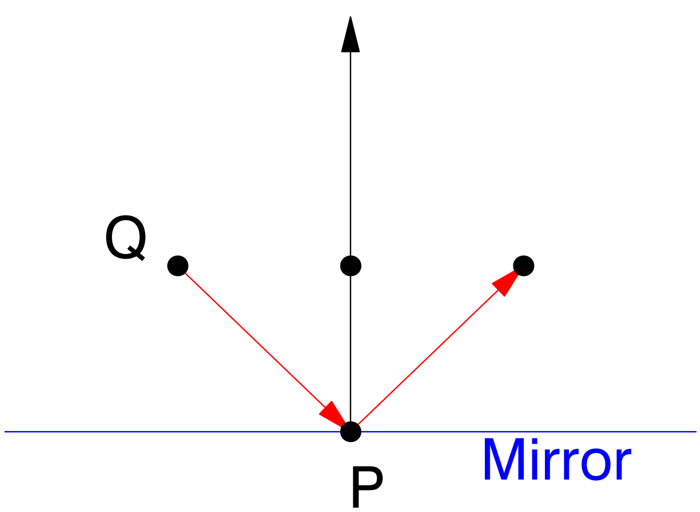
Solution:
a) $\vec{v}(t) = \vec{r}\,'(t) = \langle 1/t, 2, 2t \rangle$.
$\vec{v}(1) = \langle 1, 2, 2 \rangle$
$\vec{a}(t) = \vec{r}\,''(t) = \langle -1/t^2, 0, 2 \rangle$.
$\vec{a}(1) = \langle -1, 0, 2 \rangle$.
b) $\int_1^2 \sqrt{1/t^2 + 4 + 4t^2}\, dt = \int_1^2 1/t + 2t\, dt = \log(t) + t^2|_1^2 = \log(2) + 3$.

Problem 8) (10 points)

A planar mirror in space contains the point $P = (4, 1, 5)$ and is perpendicular to the vector $\vec{n} = \langle 1, 2, -3 \rangle$. The light ray $\vec{QP} = \vec{v} = \langle -3, 1, -2 \rangle$ with source $Q = (7, 0, 7)$ hits the mirror plane at the point P.

a) (4 points) Compute the projection $\vec{u} = \vec{\mathrm{P}}_{\vec{n}}(\vec{v})$ of $\vec{v}$ onto $\vec{n}$.

b) (6 points) Identify $\vec{u}$ in the figure and use it to find a vector parallel to the reflected ray.



Solution:
a) $P_{\vec{n}}(\vec{v}) = \frac{(\vec{n}\cdot\vec{v})}{|\vec{n}|^2}\vec{n} = (\langle 1,2,-3\rangle \cdot \langle -3,1,-2\rangle)/14\vec{n} = (5/14)\langle 1,2,-3\rangle$.
b) With $\vec{u}$ we can get the reflected vector $\vec{w}$ because $\vec{w}-\vec{v} = -2\vec{u}$ so that $\vec{w} = \vec{v} - 2\vec{u}$. Note that $\vec{u}$ points down towards the mirror.

Problem 9) (10 points)

We know the acceleration $\vec{r}\,''(t) = \langle 2,1,3\rangle + t\langle 1,-1,1\rangle$ and the initial position $\vec{r}(0) = \langle 0,0,0\rangle$ and initial velocity $\vec{r}\,'(0) = \langle 11,7,0\rangle$ of an unknown curve $\vec{r}(t)$. Find $\vec{r}(6)$.

Solution:

$$\vec{r}\,'(t) = \int_0^t \langle 2+t, 1-t, 3+t\rangle\; dt + \vec{r}\,'(0) = \langle 11+2t+t^2/2, 7+t-t^2/2, 3t+t^2/2\rangle$$

$$\vec{r}\,(t) = \int_0^t \langle 2t+t^2/2, t-t^2/2, 3t+t^2/2\rangle\; dt + \vec{r}(0) = \langle 11t+t^2+t^3/6, 7t+t^2/2-t^3/6, 3t^2/2+t^3/6\rangle$$

Plug in the time $t=6$ gives $= \boxed{\langle 138, 24, 90\rangle}$.

Problem 10) (10 points)

Intersecting the elliptic cylinder $x^2 + y^2/4 = 1$ with the plane $z = \sqrt{3}x$ gives a curve in space.

a) (3 points) Find the parametrization of the curve.

b) (3 points) Compute the unit tangent vector $\vec{T}$ to the curve at the point $(0,2,0)$.

c) (4 points) Write down the arc length integral and evaluate the arc length of the curve.

Solution:
a) With $x = \sin(t), y = 2\cos(t), z = \sqrt{3}\sin(t)$, we check $x^2 + y^2/4 = \sin^2(t) + \cos^2(t) = 1$. The parametrization is $\vec{r}(t) = \langle \sin(t), 2\cos(t), \sqrt{3}\sin(t)\rangle$.
b) Compute $\vec{r}\,'(t) = \langle \cos(t), -2\sin(t), \sqrt{3}\cos(t)\rangle$, the speed $|\vec{r}\,'(t)| = 2$ and $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,')(t)| = \langle \cos(t)/2, -\sin(t), \sqrt{3}\cos(t)/2\rangle$.
c) $|\vec{r}(t)| = 2$. The length is $\int_0^{2\pi} |\vec{r}\,'(t)|\; dt = \int_0^{2\pi} 2\; dt = \boxed{4\pi}$.