Homework 30: Greens theorem

This homework is due Wednesday, 11/19 rsp Thursday 11/20.

1 Evaluate the line integral

$$\oint_C \vec{F} \cdot d\vec{r} \ ,$$

where $\vec{F}(x, y) = \langle y + x^5 - 1, \sin(y)y^{100}\rangle$ and C consists of the line segments from $(0, 1)$ to $(0, 0)$ and from (0,0) to (1,0) and the parabola $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$.

2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$\vec{F}(x, y) = \langle y^2 \ \cos x + \sin(\sin(x)), 5x + x^2 + 2y \ \sin x + \sin(\sin(y))\rangle$

where C is the triangular path from (0,0) to (2,6) to (2,0) to (0,0). Watch the orientation of the curve! b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F}(x, y) = \langle y - \ln(x^2 + y^2), 2\tan^{-1}(y/x)\rangle$$

C is the circle $(x-2)^2 + (y-3)^2 = 1$ oriented counterclockwise.

3 A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 2\cos^3(t), 2\sin^3(t)\rangle, 0 \leq t \leq 2\pi \ .$$

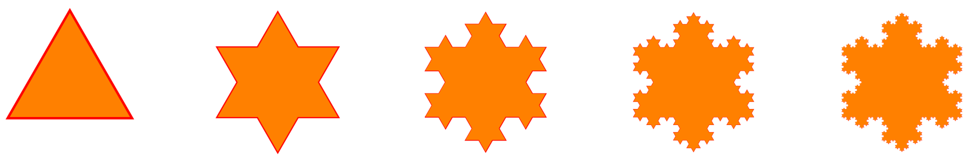
We can not do that directly. Guess which theorem to use!

4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle x^2 + y, 3x - y^2\rangle$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise. (You might have seen in class the computation of the area of an ellipse using Green, you can use that).

5 The coolest Green problem! Use Green's Theorem to evaluate

$$\int_C \langle \sin(\sqrt{1+x^3}), 7x \rangle \, d\vec{r} \, ,$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length 1/3 added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length 1/9 added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length 1/27 added and $K(4)$ is $K(3)$ with $3 * 4^3$ triangles of length 1/81 added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26...$ which is between 1 and 2.



Main points

The **curl** of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \, .$$

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field and R is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \int\int_G \text{curl}(F) \, dxdy \, .$$