Homework 3: Cross product

This homework is due Monday, 9/15 rsp Tuesday 9/16.

1 a) Find the cross product $\vec{a} \times \vec{b}$ for $\vec{a} = \langle 1, 1, 7 \rangle, \vec{b} = \langle 2, -2, 4 \rangle$ and check that it is orthogonal to both $\vec{a}$ and $\vec{b}$.
b) If we write $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$. Compute $\vec{j} \times \vec{i}, \vec{i} \times (\vec{j} \times \vec{i}), \vec{j} \times (\vec{i} \times (\vec{j} \times \vec{i}))$ etc. Can you see a pattern?

2 a) Find a nonzero vector orthogonal to the plane through the points $P = (-2, 3, 1), Q = (1, 5, 2), R = (4, 3, -1)$.
b) Find the area of the triangle PQR.

3 Let $\vec{v} = \langle 0, 5, 0 \rangle$ and consider a general vector $\vec{u} = \langle 2\cos(\alpha), 2\sin(\alpha), 0 \rangle$ of length 2 that starts at the origin and is located in the xy-plane. Find the maximum and minimum values of the length of the vector $\vec{u} \times \vec{v}$ as α rotates from 0 to 2π.

4 a) Use the triple scalar product to determine whether the points $A = (1, 1, 2), B = (3, -1, 6), C = (5, 2, 0)$ and $D = (1, -4, 12)$ lie in the same plane.
b) Write down a general formula for the volume of the tetrahedron with vertices $A = (0, 0, 0), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3), D = (d_1, d_2, d_3)$. (First establish that a tetrahedron is a fraction of the volume of a parallelepiped).

5 Verify the following formula for the **vector triple product** $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ by brute force.

Main definitions

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ in the plane is the scalar $v_1 w_2 - v_2 w_1 = \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle \ .$$

To remember it we write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. We will see the formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$. The **orientation** of three vectors is defined as the sign of $[\vec{u}, \vec{v}, \vec{w}]$. Its positivity can be established with the right hand rule.