Homework 29: Theorem of line integrals

This homework is due Monday, 11/17 rsp Tuesday 11/18.

1 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x,y) = \langle x^2, y^2 \rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.

2 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x,y) = \langle y^2/(1+x^2), 2y\arctan(x) \rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = \langle t^2, 2t \rangle$ with $0 \leq t \leq 1$.

3 a) Find the work done by the force field $\vec{F}$ in moving an object from $P = (0,1)$ to $Q = (2,0)$.

$$\vec{F}(x,y) = \langle e^{-y}, -xe^{-y} \rangle; \ .$$

b) You swim in a field $\vec{F} = \langle x^8, \sin(y) \rangle$ along a path $\vec{r}(t) = \langle t, t^2 + \sin(\sin(\pi t)) \rangle$ from $t = 0$ to $t = 2$. Find the energy you have spent.

4 a) Verify that if $\vec{F} = \langle P, Q, R \rangle$ is conservative, then

$$P_y = Q_x, P_z = R_x, Q_z = R_y \ .$$

b) Is $\langle x^5 y, xy^2, zx \rangle$ conservative? If yes, find f such that $\vec{F} = \nabla f$, if not, give a reason.

5 a) Show that the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

with $F(x,y,z) = \langle y, x, xyz \rangle$ is not conservative by using problem 4).
b) Find two different curves from $(0,0,0)$ to $(1,1,0)$ for which the line integral is different.

Main points

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

This theorem can be used to dramatically simplify the computation of a line integral.

A region R is called **simply connected** if every closed loop in R can be pulled together to a point within R.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:
Gradient field $\leftrightarrow$ Conservative $\leftrightarrow$ Closed loop property
In simply connected open regions, these three properties are all equivalent to being irrotational: $\mathrm{curl}(\vec{\mathrm{F}}) = \mathrm{Q_x} - \mathrm{P_y}$.