Homework 28: Line integrals

This homework is due Friday, 11/14 rsp Tuesday 11/17.

1 a) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if

$$\vec{F}(x,y,z) = \langle x+y, y-z, z^2 \rangle$$

and $\vec{r}(t) = \langle t^2, t^3, t^2 \rangle$ with $0 \leq t \leq 1$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if

$$\vec{F}(x,y,z) = \langle z, y, -x \rangle$$

and

$$\vec{r}(t) = \langle t, \sin(t), \cos(t) \rangle,\ 0 \leq t \leq \pi\ .$$

2 An electric current I produces a magnetic field $\vec{B}$ whose flow lines are circles circling the wire. Let $C : \langle r\cos(t), r\sin(t), 0 \rangle$. Ampèrs law is

$$\int_C \vec{B} \cdot d\vec{r} = \mu_0 I\ ,$$

where μ_0 is a constant called permeability. Show that the magnitude $B(r) = |\vec{B}|$ of the magnetic field at a distance r from the center of the wire is $B = \frac{\mu_0 I}{2\pi r}$.

3 Determine from each of the following cases, whether or not $\vec{F}$ is conservative or not. If it is, find a function f such that $\vec{F} = \nabla f$.

a) $\vec{F}(x,y) = \langle e^x \sin(y), e^x \cos(y) \rangle$
b) $\vec{F}(x,y) = \langle 3x^2 - 2y^2), 4xy + 3 \rangle$
c) $\vec{F}(x,y,z) = \langle x+y, y+x, z^5 - \sin(z) \rangle$

4 Evaluate the line integral $\int_C \langle 1 - ye^{-x}, e^{-x} \rangle \cdot d\vec{r}$, where C is the path $r(\vec{t}) = \langle t, 1 + t + \sin(\sin(\pi t)) \rangle$ and t is from 0 to 1.

5 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
a) $\{(x, y) \mid 0 < y < 3\}$, b) $\{(x, y) \mid 1 < |x| < 2\}$
c) $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$ d) $\{(x, y) \mid (x, y) \neq (1, 2)\}$
e) $\{(x, y, z) \mid (x, y, z) \neq (1, 2, 3)\}$

Main definitions

If $\vec{F}$ is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is called the **line integral** of $\vec{F}$ along the curve C.

A vector field is **conservative** in a region R if the line integral from A to B is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. It is **irrotational** if $\mathrm{curl}(F) = Q_x - P_y$ is zero everywhere in R.

A subset G of the plane is **open** if every point (x, y) in G is contained in a mall disc centered at (x, y) which is also in G. A subset is **connected**, if one can connect any two points in G with a curve inside G. A subset G is **simply connected** if every closed curve in G can be pulled together to a point within G.

Clairaut test: Zero curl is necessary for a gradient field.