

## Homework 25: Triple integrals

This homework is due Wednesday, 11/5 resp Tuesday 11/11.

1 Evaluate the iterated integral

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} 12 dx dz dy .$$

**Solution:**

Compute:

$$\begin{aligned} \int_0^2 \int_0^{2-y} \int_0^{4-y^2} 12 dx dz dy &= \int_0^2 \int_0^{2-y} 12(4 - y^2) dz dy \\ &= 12 \int_0^2 (4 - y^2)(2 - y) dy \\ &= 80. \end{aligned}$$

2 Evaluate the triple integral

$$\int \int \int_E yz \cos(x^5) dV ,$$

where

$$E = \{(x, y, z) \mid 0 \leq x \leq \pi/2, 0 \leq y \leq x, x \leq z \leq 2x\} .$$

**Solution:**

By the given description of  $E$ , we can write the triple integral as an iterated integral:

$$\iiint_E yz \cos(x^5) dV = \int_0^{\pi/2} \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx.$$

We integrate this in the usual way:

$$\begin{aligned} \int_0^{\pi/2} \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx &= \int_0^{\pi/2} \int_0^x y \cos(x^5) \cdot \left( \frac{z^2}{2} \Big|_{z=x}^{2x} \right) dy dx \\ &= \frac{3}{2} \int_0^{\pi/2} \int_0^x x^2 y \cos(x^5) dy dx \\ &= \frac{3}{2} \int_0^{\pi/2} x^2 \cos(x^5) \cdot \left( \frac{y^2}{2} \Big|_{y=0}^x \right) dx \\ &= \frac{3}{4} \int_0^{\pi/2} x^4 \cos(x^5) dx \\ &= \frac{3}{20} \sin(x^5) \Big|_{x=0}^{\pi/2} \\ &= \frac{3}{20} \sin \left( \left( \frac{\pi}{2} \right)^5 \right). \end{aligned}$$

3 Evaluate the triple integral

$$\int \int \int_E xy dV ,$$

where  $E$  is bounded by the parabolic cylinders  $y = 3x^2$  and  $x = 3y^2$  and the planes  $z = 0$  and  $z = x + y$ .

## Solution:

One way to describe the region  $E$  is as

$$E = \{(x, y, z) : 0 \leq z \leq x + y, (x, y) \in D\}$$

where  $D$  is the region in the  $xy$ -plane bounded by the two parabolas  $y = 3x^2$  and  $x = 3y^2$ . Thus we can write our integral as either

$$\iiint_E xy \, dV = \iint_D \int_0^{x+y} xy \, dz \, dA = \int_0^{1/3} \int_{3x^2}^{\sqrt{\frac{x}{3}}} \int_0^{x+y} xy \, dz \, dy \, dx$$

or

$$\int_0^{1/3} \int_{3y^2}^{\sqrt{\frac{y}{3}}} \int_0^{x+y} xy \, dz \, dx \, dy$$

These last two iterated integrals are equivalent (because of the symmetry in  $x$  and  $y$ ), and we'll compute only the first:

$$\begin{aligned} \iiint_E xy \, dV &= \int_0^{1/3} \int_{3x^2}^{\sqrt{\frac{x}{3}}} \int_0^{x+y} xy \, dz \, dy \, dx \\ &= \int_0^{1/3} \int_{3x^2}^{\sqrt{\frac{x}{3}}} xy(x+y) \, dy \, dx \\ &= \int_0^{1/3} \int_{3x^2}^{\sqrt{\frac{x}{3}}} (x^2y + xy^2) \, dy \, dx \\ &= \int_0^{1/3} \left( x^2 \cdot \frac{y^2}{2} + x \cdot \frac{y^3}{3} \right) \Big|_{y=3x^2}^{\sqrt{\frac{x}{3}}} dx \\ &= \int_0^{1/3} \left( \frac{x^3}{6} - \frac{9x^6}{2} + \frac{x^{5/2}}{3^{5/2}} - 9x^7 \right) dx \\ &= \left( \frac{x^4}{24} - \frac{9x^7}{14} + \frac{2x^{7/2}}{7 \cdot 3^{5/2}} - \frac{9x^8}{8} \right) \Big|_{x=0}^{1/3} \\ &= \frac{1}{2268}. \end{aligned}$$

- 4 Use a triple integral to find the volume of the given solid enclosed by the paraboloid  $x = y^2 + z^2$  and the plane  $x = 25$ .

**Solution:**

The paraboloid  $x = y^2 + z^2$  intersects the plane  $x = 25$  in the circle  $y^2 + z^2 = 25$ . Thus,

$$E = \{(x, y, z) \mid y^2 + z^2 \leq x \leq 25, (y, z) \in D\}$$

where  $D = \{(y, z) \mid y^2 + z^2 \leq 25\}$ . Then using polar coordinates  $y = r \cos \theta$  and  $z = r \sin \theta$ , we have

$$\begin{aligned} \iiint_D \left( \int_{y^2+z^2}^{25} dx \right) dA &= \iint_0^{2\pi} \int_0^5 \int_{r^2}^{25} r \, dx \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^5 (25 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^5 (25r - r^3) \, dr \\ &= 2\pi \left( \frac{25r^2}{2} - \frac{r^4}{4} \right) \Big|_{r=0}^5 \\ &= \frac{625\pi}{2}. \end{aligned}$$

5 Find the moment of inertia

$$\int \int \int_E (x^2 + y^2) \, dx \, dy \, dz$$

about the  $z$ -axis of the solid cone  $E : \sqrt{x^2 + y^2} \leq z \leq 10$ .

**Solution:**

The moment of inertia is given by

$$\begin{aligned}\iint\int_E (x^2 + y^2) \, dx \, dy \, dz &= \iint_{x^2 + y^2 \leq 10^2} \left[ \int_{\sqrt{x^2 + y^2}}^{10} (x^2 + y^2) \, dz \right] dA \\ &= \iint_{x^2 + y^2 \leq 10^2} (x^2 + y^2) (10 - \sqrt{x^2 + y^2}) \, dA \\ &= \int_0^{2\pi} \int_0^{10} r^2 (10 - r) r \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{10} (10r^3 - r^4) \, dr \\ &= (2\pi) \left( \frac{10}{4} r^4 - \frac{1}{5} r^5 \right) \Big|_{r=0}^{10} \\ &= 10,000\pi.\end{aligned}$$

## Main definitions

If  $f(x, y, z)$  is a function of three variables and  $E$  is a **solid region** in space, then  $\int \int \int_E f(x, y, z) dV$  is defined as the  $n \rightarrow \infty$  limit of the Riemann sum

$$\frac{1}{n^3} \sum_{\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) \in E} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right).$$

As in two dimensions,  $dA = dx dy$  is the symbol for a small area,  $dV = dx dy dz$  reads as a small volume.

If  $f(x, y, z) = 1$  then  $\int \int \int_E 1 dx dy dz$  is the volume of the solid

In multivariable calculus we often encounter situations, where the triple integral is reduced to a double integral

$$\int \int_R \left[ \int_{g(x,y)}^{h(x,y)} f(x, y, z) dz \right] dx dy .$$

For example, if  $g(x, y) = 0$  and  $f(x, y, z) = 1$ , then

$$\int \int_R \left[ \int_0^{h(x,y)} 1 dz \right] dx dy = \int \int_R h(x, y) dx dy$$

is the signed volume of the solid under the graph of  $h$ .