Homework 16: Tangent lines and planes

This homework is due Friday, 10/17 rsp Thursday 10/16.

1 The equation $f(x,y) = x^4 y + 5xy^5 = 26$ defines a curve in the xy-plane. Find the tangent line $ax + by = d$ to the curve at $(2,1)$ by computing the gradient $\nabla f(x,y) = \langle a, b \rangle$, and then plugging in the point to get the constant d.

2 a) Find an equation of the tangent plane to the surface $z = 3(x-1)^2 + 2(y+3)^2 + 7$ at the point $(2,-2,12)$.
b) Find an equation of the tangent plane to the surface

$$z = \log(x - 2y)$$

(with log = ln as natural log as usual) at the point $(3,1,0)$.

3 a) Find an equation of the tangent plane to the parametric surface

$$\vec{r}(u,v) = \langle u^2, v^2, uv \rangle$$

at the point $(u,v) = (1,1)$.
b) The surface satisfies the equation $x * y - z^2 = 0$. Find the tangent plane to this surface at the same point $(x,y,z) = (1,1,1)$ by computing the gradient.

4 Find an equation of the tangent plane and the normal line to the surface $x - z - 4\arctan(yz) = 0$ through the point $(1+\pi, 1, 1)$.

5 Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1,1,2)$ meaning that they have the same tangent plane at that point.

Main definitions

The **gradient** in two dimensions is $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$. In three dimensions, it is $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$.

From the chain rule, we can deduce:

Gradient theorem: Gradients are orthogonal to level curves and level surfaces.

The tangent line through (x_0, y_0) to a level curve $f(x,y) = c$ is $ax + by = d$, where $\nabla f(x_0, y_0) = \langle a, b\rangle$ and d is obtained by plugging in the point.

The tangent plane through (x_0, y_0, z_0) to a level surface $f(x,y,z) = c$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = \langle a, b, c\rangle$ and d is obtained by plugging in the point.

We can compute tangent planes also for parametrized surfaces $\vec{r}(u,v)$ because the vectors $\vec{r}_u, \vec{r}_v$ are velocity vectors of grid curves and so tangent to the surface. Get the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle a, b, c\rangle$ and then get $ax + by + cz = d$, where d is obtained by plugging in the point $\vec{r}(u_0, v_0)$. Here is how to plot parametric surfaces:

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ParametricPlot3D[{u,v^2,u v},{u,0,1},{v,0,1}]
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