Homework 13: Partial differential equations

This homework is due Wednesday, 10/8 rsp Friday 10/10.

1 Determine which of the following functions solves the **Laplace's equation** $u_{xx} + u_{yy} = 0$.
a) $u = 5x^2 + 5y^2$ b) $u = 7x^2 - 7y^2$ c) $u = x^3 + 3xy^2$
d) $u = \log \sqrt{x^2 + y^2}$ e) $u = e^{-x} \cos y - e^{-y} \cos x$
As usual, $\log = \ln$ denotes the natural log.

2 Show that each of the following functions is a solution of the **wave equation** $u_{tt} = u_{xx}$.
a) $u = \sin(kx)\sin(kt)$ b) $u = \left(\frac{t}{t^2-x^2}\right)$
c) $u = (x-t)^6 + (x+t)^6$ d) $u = \sin(x-t) + \log(x+t)$

3 Show that the **Cobb-Douglas** production function $P = L^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P\ .$$

The constants α and β are fixed. L is labor and K is cost.

4 a) Run Mathematica code for the one dimensional wave equation. The example code is given. We want to know the value of $u(t, x)$ at $t = 0.4$ and $x = 0.3$.
b) Run Mathematica code for the two dimensional wave equation. The example code is given. Plot the graph of the function $u(t, 0.3, 0.4)$ from $t = 0$ to $t = 1$. You can either print out your output or copy what you see on the screen.

5 The partial differential equation

$$f_t + ff_x = f_{xx}$$

called **Burgers equation** describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which are used to describe the weather. Use Mathematica to verify that

$$f(t,x)\frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}}e^{-\frac{x^2}{4t}}+1}$$

solves the Burgers equation.

Use of Mathematica

Part of the homework is also to install and run Mathematica. You can use this software to make tough computations. You can use the software to do the problems above. Here is an example. After entering, type return while holding down the shift key:

```
f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[ D[f[t,x],t] == D[f[t,x],{x,2}]]
```

You have verified that the function

$$\frac{1}{\sqrt{t}}e^{-x^2/(4t)}$$

satisfies the heat equation. As any real programming language, Mathematica is particular about syntax. Watch brackets, capitalization, double equal signs ==!

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**.

Here is Mathematica code for an example of the two dimensional wave equation:

```
A=Rectangle[{0,0},{1,1}]; Clear[t,x,y];
f[x_,y_]:=Sin[2 Pi x] Abs[Sin[3 Pi y]];
g[x_,y_]:=3 Sin[Pi x] Sin[Pi y];
U=NDSolveValue[{D[u[t,x,y],{t,2}]
 -Inactive[Laplacian][u[t,x,y],{x,y}]==0,
   u[0,x,y] == f[x,y],
   Derivative[1,0,0][u][0,x,y]==g[x,y],
   DirichletCondition[u[t,x,y] ==0,True]},
   u,{t, 0, 2 Pi}, {x,y} \[Element] A];
Plot3D[U[4,x,y],{x,0,1},{y,0,1}]
Animate[ContourPlot[U[t,x,y],
    {x,0,1},{y,0,1}],{t,0,2Pi}]
```

Here is an example for Mathematica code for the one dimensional wave equation:

```
f[x_]:=Sin[Pi 7x];
g[x_]:=5 Sin[5 Pi x];
U = NDSolveValue[
   {D[u[t,x],{t,2}]-D[u[t,x],{x,2}]==0,
   u[0,x]==f[x],
   Derivative[1,0][u][0,x] == g[x],
   DirichletCondition[u[t,x]==f[0],x==0],
   DirichletCondition[u[t,x]==f[1],x==1]},
   u,{t,0,1},{x,0,1}];
Animate[Plot[U[t,x],{x,0,1},
   PlotRange->{-2,2}],{t,0,1,0.01}]
Plot[U[t,0.5], {t, 0, 1}]
```