Homework 12: Partial derivatives

This homework is due Monday, 10/8 rsp Tuesday 10/9.

1 If $f(x,y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1,0)$ and $f_y(1,0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

2 Find the partial derivatives $f_x(x,y)$, $f_y(x,y)$ of the function $f(x,y) = 5x^y$.

3 Find the first partial derivatives $f_x(x,y)$, $f_y(x,y)$ of the function

$$f(x,y) = \int_y^x \sin(t^2) \; dt \; .$$

4 Verify that the function

$$u(x,t) = e^{-\alpha^2 k^2 t} \sin(kx)$$

is a solution of the heat conduction equation $u_t(x,t) = \alpha^2 u_{xx}(x,t)$. Here k, α are constants.

5 Verify that the function

$$u(x,y,z) = 1/\sqrt{x^2 + y^2 + z^2}$$

is a solution of the three dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

Main definitions

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x}f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly. We also write $f_x(x,y) = \frac{\partial}{\partial x}f(x,y)$. and $f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f$.

Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**. Here are examples:

1 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound.

2 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic.

3 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane.

4 The **advection equation** $f_t = f_x$ is used to model transport in a wire.

5 The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics.

6 The **Burgers equation** $f_t + ff_x = f_{xx}$ describes waves at the beach which break.