Homework 11: Functions and Continuity

This homework is due Friday, 10/5 rsp Tuesday 10/9.

1 Which of the following functions are continuous everywhere on the real line? No reasoning is required. This is a hit or miss question, each question one point. The log is the natural log.

a) $\log(|x|)$
b) $1/\log(|x|)$
c) $1/\log((0.5|\sin(x)|)$
d) $\sin(\log|x|)$
e) $1/\log(2+|x|)$
f) $\log(\exp(x))$
g) $\exp(1/x)$
h) $1/\exp(1/x)$
i) $\exp(1/x)$
j) $\exp(\log|x|)$

2 Investigate whether the following functions are continuous at $(0,0)$. If the limit $(x,y) \to (0,0)$ exists, tell what the limit is.
a) $f(x,y) = x^2/(x^3+y^3)$.
b) $f(x,y) = x^4/(x^3+y^3)$.
It can be helpful to use polar coordinates. You have to give some reasoning here.

3 Find the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists or show that the limit does not exist
a) $f(x,y) = \frac{6x^4y}{2x^5+y^5}$
b) $f(x,y) = \frac{x^6-y^6}{(x^2+y^2)^2}$
Also here is some reasoning required.

4 Determine the set of points where the following function is continuous

$$f(x,y) = \frac{e^x+e^y}{e^{xy}-1} .$$

5 Use polar coordinates to find the limit $(x, y) \to (0, 0)$ of the function

$$f(x, y) = \sin(x^2 + y^2) \log(x^2 + y^2) \ ,$$

where $\log(x) = \ln(x)$ is the natural logarithm.

Main definitions

$f(x, y)$ with domain R is **continuous at a point** $(a, b) \in R$ if $f(x, y) \to f(a, b)$ whenever $(x, y) \to (a, b)$. It is continuous if its domain of definition can be extended to the entire plane and be continuous there. With this notation, $f(x, y) = y(x^2 - 1)/(x - 1)$ is continuous; it is at first not defined at $x = 1$ but has an extension $f(x, y) = y(x + 1)$. Similarly, the function $f(x, y) = \sin(x^2 + y^2)/(x^2 + y^2)$ is continuous everywhere as one can see with Hopital for polar coordinates. Discontinuity types in one dimensions are **jump discontinuities** like $f(x) = \text{sign}(x)$ or **poles** like $f(x) = 1/x$ or **oscillations** like $f(x) = \sin(1/x)$. These three prototypes can occur together in examples like $1/\sin(1/x))$ or $\arctan(1/\sin(1/x))$. Many questions about continuity of a function $f(x, y)$ of two variables can be answered when writing the function in polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ near the point in question. For $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ for example, the function becomes (just fill in $x = r\cos(\theta), y = r\sin(\theta)$), in polar coordinates $f(r, \theta) = \cos(2\theta)$. The value of the function depends only on the angle. Arbitrarily close to $(0, 0)$, the function takes any values between -1 and 1.