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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) **T** F For any two vectors $\vec{v}$ and $\vec{w}$ one has $\text{proj}_{\vec{v}}(\vec{v} \times \vec{w}) = \vec{0}$.

Solution:
The projection of a vector $\vec{v}$ onto a vector $\vec{w}$ which is perpendicular to $\vec{v}$ is zero.

2) T **F** Any parameterized surface S is either the graph of a function $f(x, y)$ or a surface of revolution.

Solution:
A counter example is an asymetric ellipsoid. It is neither a graph, nor a surface of revolution.

3) T **F** If the directional derivative $D_{\vec{v}}(f)$ of f into the direction of a unit vector $\vec{v}$ is zero, then $\vec{v}$ is perpendicular to the level curve of f.

Solution:
It is either tangent to the level curve or at a critical point.

4) **T** F The linearization $L(x, y)$ of $f(x, y) = 5x - 100y$ at $(0, 0)$ satisfies $L(x, y) = 5x - 100y$.

Solution:
The linearization of any linear function at $(0, 0)$ is the function itself.

5) **T** F If a parameterized curve $\vec{r}(t)$ intersects a surface $\{f = c\}$ at a right angle, then at the point of intersection we have $\nabla f(\vec{r}(t)) \times \vec{r}\,'(t) = 0$.

Solution:
This is clear, once you know what the question means. The conditon $\nabla f(\vec{r}(t)) \times \vec{r}\,'(t) = 0$ means that the velocity vector $\vec{r}\,'(t)$ is parallel to the gradient vector, which means that it is perpendicular to the level surface.

6) **T** F The curvature of the curve $\vec{r}(t) = \langle \cos(3t^2), \sin(6t^2) \rangle$ at the point $\vec{r}(1)$ is larger than the curvature of the curve $\vec{r}(t) = \langle 2\cos(3t), 2\sin(6t) \rangle$ at the point $\vec{r}(1)$.

Solution:
While curvature is independent of the parameterization of the curve, the two circles have also a different radius. The second curve has twice the radius, so half the curvature.

7) | T | **F** | At every point (x, y, z) on the hyperboloid $4x^2 - y^2 + z^2 = 10$, the vector $\langle 4x, -y, z \rangle$ is parallel to the hyperboloid.

Solution:
It is parallel to the gradient which is perpendicular to the level surface. If "parallel" were replaced by "normal" it would be true.

8) | **T** | F | The set $\{\phi = \pi/2, \theta = \pi/2\}$ in spherical coordinates is the positive y-axis.

Solution:
$\phi = \pi/2$ forces the vector to be on the xy-plane. The angle $\theta = \pi/2$ confines it to the y-axes.

9) | T | **F** | The integral $\int_0^1 \int_0^{2\pi} r^2 \sin(\theta) \ d\theta \, dr$ is equal to the area of the unit disk.

Solution:
We are using polar coordinates, not spherical coordinates in the plane. The correct integral is $\int_0^1 \int_0^{2\pi} r \ d\theta \ d$

10) | **T** | F | If three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ attached at the origin are in a common plane, then $\vec{u} \cdot ((\vec{v} + \vec{u}) \times \vec{w}) = 0$.

Solution:
The volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$ is $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot ((\vec{v} + \vec{u}) \times \vec{w})$. The volume is zero if and only if the parallelepiped is flat.

11) | T | **F** | If a function $f(x, y)$ has a local minimum at $(0, 0)$, then the discriminant D must be positive.

Solution:
False, we also can have $D = 0$ like for the function $f(x, y) = 1 - x^4 - y^4$.

12) T **F**

The integral $\int_0^1 \int_y^1 f(x,y)\, dy\, dx$ represents a double integral over a bounded region in the plane.

Solution:
The integral is not properly defined. There can be no variable in the most outer integral.

13) **T** F

The following identity is true: $\int_0^3 \int_0^x x^2\, dydx = \int_0^3 \int_y^3 x^2\, dx\, dy$

Solution:
Make a picture and draw the triangle.

14) T **F**

There is a quadric of the form $ax^2 + by^2 + cz^2 = d$, for which all three traces are hyperbola.

Solution:
Check through the list of all the quadrics. Paraboloids have at least one parabola as a trace. Hyperboloids have a circle or ellipse as a trace. Ellipsoids have ellipses as traces.

15) T **F**

The curvature of a space curve $\vec{r}(t)$ is a vector perpendicular to the acceleration vector $\vec{r}\,''(t)$.

Solution:
The curvature is a scalar, not a vector.

16) T **F**

Assume S is the outside oriented unit sphere. Let S^+ be the upper hemisphere and S^- the lower hemisphere. If $\vec{F}$ is incompressible, then the flux of $\vec{F}$ through S^+ is equal to the flux of $\vec{F}$ through S^-.

Solution:
The sum of the fluxes is 0 so that flux through the total sphere is 0.

17) **T** F

If $\vec{F}$ is defined everywhere except at the origin and $\text{curl}(\vec{\text{F}}) = \vec{0}$ everywhere, then the line integral of $\vec{F}$ around any closed path not passing through the origin is zero.

Solution:
The region without the origin is simply connected so that by Stokes theorem, one has a conservative vector field.

18) **T** F

Every vector field which satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere in $\mathbf{R}^3$ can be written as $\vec{F} = \text{grad}(f)$ for some scalar function f.

Solution:
This is a consequence of Stokes theorem: for any closed curve find a surface which has this curve as a boundary. By Stokes theorem, the line integral is zero. The field consequently has the closed loop property and is so a gradient field.

19) **T** F

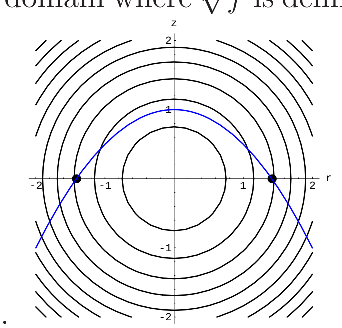
Let $\vec{F}$ be a vector field and let S be an oriented surface $\vec{r}(u,v)$. Then $20 \int \int_S \vec{F} \cdot dS = \int \int_S \vec{G} \cdot dS$, where $\vec{G} = 20\vec{F}$.

Solution:
The integral is linear.

20) T **F**

Consider the surface S given by the equation $z^2 = f(x,y)$. If $(x,y,z) = (x,y,\sqrt{f(x,y)})$ is a point on the surface with maximal distance from the origin, it is a local maximum of $g(x,y) = x^2 + y^2 + f(x,y)$.

Solution:
The point on the surface with maximal distance from the origin is the point maximizing $g(x,y)$ under the constraint $f(x,y) \geq 0$. This point could occur on the boundary $f(x,y) = 0$, in which case it doesn't need to be a critical point of $g(x,y)$. Take $f(x,y) = 1 - (x^2 - y^2)/2$. Now $g(x,y) = x^2 + y^2 + f(x,y) = 1 + (x^2 + y^2)/2$. Points like $(\sqrt{2}, 0, 0)$ have maximal distance to the origin but it is a local minimum of $g(x,y)$. The picture shows the situation in the rz plane. The extremum occurs at the boundary of the domain where $\sqrt{f}$ is defined.



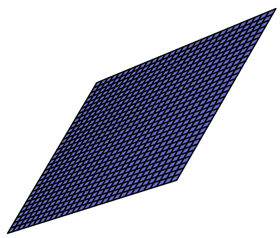
It is not a point where the distance to the origin is maximal.

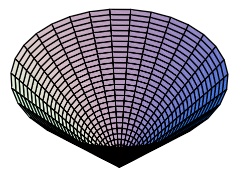
Problem 2) (10 points)

Match the parameterized surface formulas and pictures with the formulas for the implicit surfaces. No justifications are needed.

A)	$\vec{r}(u,v) = \langle 1+u, v, u+v \rangle$
B)	$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v \rangle$
C)	$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$
D)	$\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$
E)	$\vec{r}(u,v) = \langle v, \sin(u), \cos(u) \rangle$

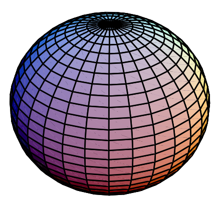
I



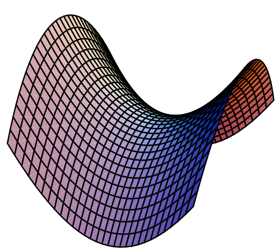


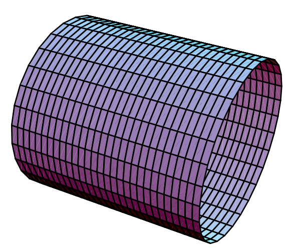
II

III



IV





V

Enter A),B),C),D),E) here	Enter I),II),III),IV),V) here	Equation
		$y^2 + z^2 = 1$
		$x + y - z = 1$
		$x^2 + y^2 + z^2 = 1$
		$x^2 + y^2 - z^2 = 0$
		$x^2 - y^2 - z = 0$

Solution:

Enter A),B),C),D),E) here	Enter I),II),III),IV),V) here	Equation
E	V	$y^2 + z^2 = 1$
A	I	$x + y - z = 1$
C	III	$x^2 + y^2 + z^2 = 1$
B	II	$x^2 + y^2 - z^2 = 0$
D	IV	$x^2 - y^2 - z = 0$

Problem 3) (10 points)

Match the formulas and theorems with their names. No justifications are needed.

N) $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$
U) $f_{xy}(x,y) = f_{yx}(y,x)$
M) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$
E) $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$
R) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
A) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$
L) $(\vec{u}\times\vec{v})\cdot\vec{w}$
F) $|\vec{v}\cdot\vec{w}| \leq |\vec{v}|\cdot|\vec{w}|$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ if $\vec{v}\cdot\vec{w} = 0$
G) $|\vec{v}+\vec{w}| \leq |\vec{v}| + |\vec{w}|$

Enter letters here	Object or theorem
	Fubini theorem
	Clairaut theorem
	vector projection
	scalar projection
	chain rule
	dot product formula
	scalar triple product
	Cauchy-Schwarz inequality
	Pythagorean theorem
	Triangle inequality

Solution:

The solution is unscambled. [Originally, we planned to make a nice anagram, but somehow, the scrambling was only done in the solutions. Here had been the cool anagram NUMERAL FIG → FUELING ARM:]

F) $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| \cdot |\vec{w}|$.
U) $f_{xy}(x, y) = f_{yx}(x, y)$.
E) $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$
L) $(\vec{u} \times \vec{v}) \cdot \vec{w}$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v} - \vec{w}|^2$ if $\vec{v} \cdot \vec{w} = 0$.
N) $\int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy$.
G) $|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$.
A) $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\alpha)$
R) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
M) $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} w$.

The actual solution to the problem posed in the exam was that each letter stayed at the height it had been before:

N) $\int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy$
U) $f_{xy}(x, y) = f_{yx}(y, x)$
M) $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$
E) $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$
R) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
A) $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\alpha)$
L) $(\vec{u} \times \vec{v}) \cdot \vec{w}$
F) $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| \cdot |\vec{w}|$
I) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v} - \vec{w}|^2$ if $\vec{v} \cdot \vec{w} = 0$
G) $|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$

Enter letters here	Object or theorem
N	Fubini theorem
U	Clairaut theorem
M	vector projection
E	scalar projection
R	chain rule
A	dot product formula
L	scalar triple product
F	Cauchy-Schwarz inequality
I	Pythagorean theorem
G	Triangle inequality

Problem 4) (10 points)

Let L be the line $\vec{r}(t) = \langle t, 0, 0 \rangle$. We are also given a point $Q = (3, 3, 0)$ in space.

a) (2 points) What is the distance $d((x, y, z), Q)$ between a general point (x, y, z) and Q?

b) (3 points) What is the distance $d((x, y, z), L)$ between the point (x, y, z) and the line L?

c) (3 points) Find the equation for the set C of all points (x, y, z) satisfying

$$d((x, y, z), Q) = d((x, y, z), L) .$$

d) (2 points) Identify the surface.

Solution:

a) $\sqrt{(x-3)^2+(y-3)^2+z^2}$.

b) The distance formula is $|\langle 1,0,0\rangle \times \langle (x,y,z)\rangle|/\sqrt{1} = |\langle -y, x-z, y\rangle| = \sqrt{y^2+z^2}$.

c) The equation $y^2+z^2 = (x-3)^2+(y-3)^2+z^2$ can be simplified to $\boxed{(x-3)^2 = 6y-9}$.

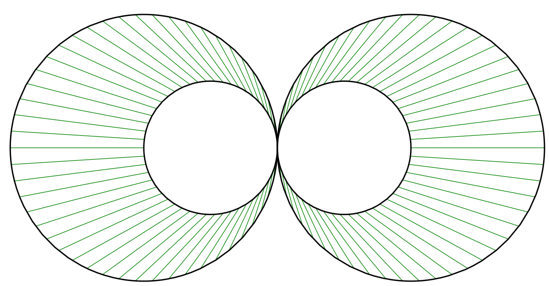
d) This is a $\boxed{\text{cylindrical paraboloid}}$. It has the same shape as $x^2 = y$ but is translated and scaled in the y direction.

Problem 5) (10 points)

Find the area of the region in the plane given in polar coordinates by

$$\{(r,\theta) \mid |\cos(\theta)| \leq r \leq 2|\cos(\theta)|, 0 \leq \theta < 2\pi\ \} .$$

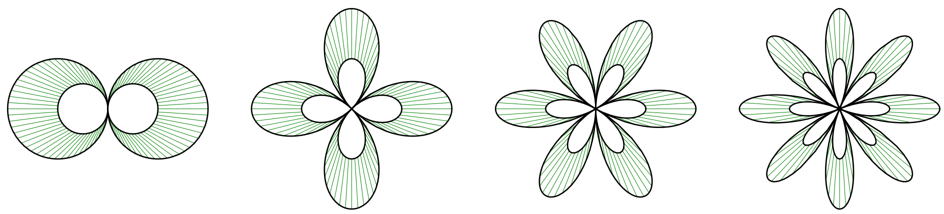
The region is the shaded part in the figure.



Solution:
$\int_0^{2\pi} \int_{|\cos(\theta)|}^{|2\cos(\theta)|} r \; dr d\theta = \int_0^{2\pi} 4\cos^2(\theta)/2 - \cos^2(\theta)/2 \; d\theta = \boxed{3\pi/2}$. The result is the same for any region

$$\{(r,\theta) \mid |\cos(n\theta)| \leq r \leq 2|\cos(n\theta)|, 0 \leq \theta < 2\pi \} .$$

If this had been a source for an error, we would not penalize it in our grading.)



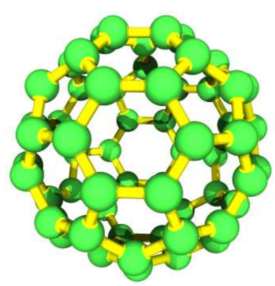
n=1 n=2 n=3 n=4

Problem 6) (10 points)

A microscopic bucky ball C60 is located on a gold surface. The surface produces the electric potential $f(x,y) = x^4 + y^4 - 2x^2 - 8y^2 + 5$.

a) (7 points) Find all critical points of f and classify them.

b) (3 points) The fullerene will settle at a global minimum of $f(x,y)$. Find the global minima of the function $f(x,y)$.



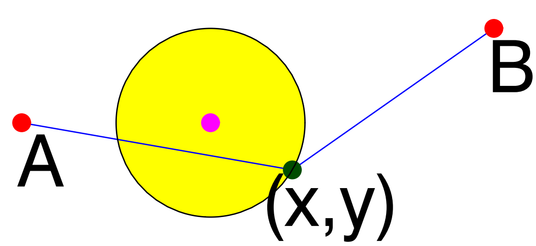
Solution:
a) The gradient of $f(x,y)$ is $\nabla f(x,y) = \langle 4x^3 - 4x, 4y^3 - 16y \rangle$. It is zero if $x = 0, 1, -1$ and $y = 0, 2, -2$. There are 9 critical points. The discriminant is $D = 16(3x^2 - 1)(3y^2 - 4)$ and and $X = f_{xx} = 12x^2 - 4 = 4(3x^2 - 1)$. Note that f_{xx} is negative if $x = 0$ and positive in all other cases.

point	D	f_{xx}	nature	value
(-1, -2)	256	8	min	-12
(-1, 0)	-128	8	saddle	4
(-1, 2)	256	8	min	-12
(0, -2)	-128	-4	saddle	-11
(0, 0)	64	-4	max	5
(0, 2)	-128	-4	saddle	-11
(1, -2)	256	8	min	-12
(1, 0)	-128	8	saddle	4
(1, 2)	256	8	min	-12

b) The points of global minima are $(1,2), (1,-2), (-1,2), (-1,-2)$. The global minimum value is $\boxed{-12}$.

Problem 7) (10 points)

A circular wheel with boundary $g(x,y) = x^2 + y^2 = 1$ has the boundary point (x,y) connected to two points $A = (-2,0)$ and $B = (3,1)$ by rubber bands. The potential energy at position (x,y) is by Hooks law equal to $f(x,y) = (x+2)^2 + y^2 + (x-3)^2 + (y-1)^2$, the sum of the squares of the distances to A and B. Our goal is to find the position (x,y) for which the energy is minimal. To find this position for which the wheel is at rest, minimize $f(x,y)$ under the constraint $g(x,y) = 1$.



Solution:
The Lagrange equations

$$\begin{aligned} 2(x+2)+2(x-3) &= \lambda 2x \\ 2y+2(y-1) &= \lambda 2y \\ x^2+y^2 &= 1 \end{aligned}$$

simplify to

$$\begin{aligned} 2x-1 &= \lambda x \\ 2y-1 &= \lambda y \\ x^2+y^2 &= 1 \end{aligned}$$

Dividing the first equation by the second gives $x = y$. Plugging this into the third equation gives $x = \pm 1/\sqrt{2}, y = \pm 1/\sqrt{2}$. For $(x,y) = (\sqrt{2},\sqrt{2})$ we have $f(x,y) = 16 - 2\sqrt{2}$ and for $(x,y) = (-\sqrt{2},-\sqrt{2})$ we have $f(x,y) = 16 + 2\sqrt{2}$. The minimum is at $\boxed{(\sqrt{2},\sqrt{2})/2}$.

Problem 8) (10 points)

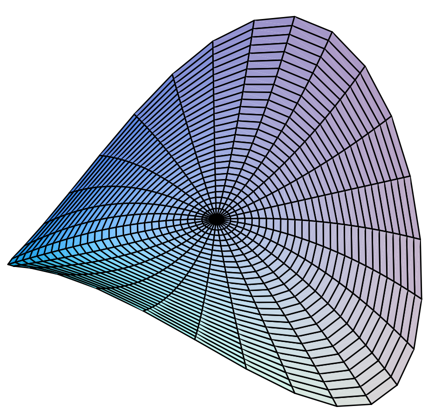
a) (5 points) Find the surface area of the parameterized surface

$$\vec{r}(u,v) = \langle u-v, u+v, uv \rangle$$

with $u^2+v^2 \leq 1$.

b) (3 points) Find an implicit equation $g(x,y,z) = 0$ for this surface.

c) (2 points) What is the name of the surface?



Solution:
a) $\vec{r}_u \times \vec{r}_v = \langle u-v, -u-v, 2\rangle$ and $|\vec{r}_u \times \vec{r}_v = \sqrt{4+2(u^2+v^2)}$. So, $\int\int |\vec{r}_u \times \vec{r}_v|\ dudv = \int_0^1 \int_0^{2\pi} \sqrt{2}\sqrt{2+r^2} r\ d\theta dr = \sqrt{2}2\pi(\sqrt{3}-2\sqrt{2}/3)$.
The answer is $\boxed{(\pi/3)(6^{3/2}-4^{3/2})}$.
b) We try to combine the functions for x, y, z so that we get something simpler. With linear combinations like $x+y, x-z$ etc, we don't get anywhere. But if we compare x^2 and y^2 we see that we can combine it to get a multiple of z. The answer is $\boxed{y^2 - x^2 = 4z}$.
c) This is a $\boxed{\text{hyperbolic paraboloid}}$, also known under the name "saddle". Some eat it as "pringles".

Problem 9) (10 points)

a) (4 points) Find the tangent plane to the surface $f(x, y, z) = zx^5 + y^5 - z^5 = 1$ at the point $(1, 1, 1)$.

b) (3 points) Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at at the point $(1, 1, 1)$.

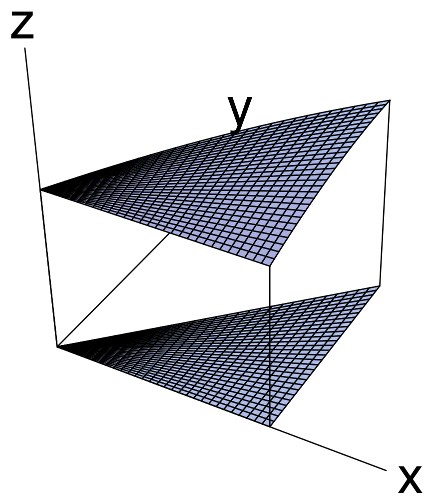
c) (3 points) Near the point $(1, 1, 1)$, the surface can be written as a graph $z = g(x, y)$. Find the partial derivative $g_x(1, 1)$.

Solution:
a) The gradient at a general point is $\nabla f(x, y, z) = \langle 5x^4z, 5y^4, x^5 - 5z^4\rangle$.
So, the gradient at the point $(1, 1, 1)$ is equal to $\nabla f(1, 1, 1) = \langle 5, 5, -4\rangle = \langle a, b, c\rangle$. The equation of the plane is $ax + by + cz = d$ where the constant d is obtained by plugging in plugging in one point. Here $\boxed{5x + 5y - 4z = 6}$.
b) The linearization is defined as $L(x, y, z) = f(1, 1, 1) + f_x(1, 1, 1)(x-1) + f_y(1, 1, 1)(y-1) + f_z(1, 1, 1)(z-1) = 1 + 5(x-1) + 5(y-1) - 4(z-1)$. This can be simplified to $\boxed{L(x, y, z) = 5x + 5y - 4z - 5}$.
c) The implicit differentiation formula derived from the chain rule gives $g_x(1, 1) = -f_x(1, 1, 1)/f_z(1, 1, 1) = \boxed{5/4}$. (This can also be seen as the slope of the xz-trace of the tangent plane.)

Problem 10) (10 points)

A tower E with base $0 \leq x \leq 1, 0 \leq y \leq x$ has a roof $f(x, y) = \sin(1-y)/(1-y)$. Find the volume of this solid. The solid is given in formulas by

$$E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le f(x, y)\ \} .$$



Solution:
The type I integral

$$\int_0^1 \int_0^x \frac{\sin(1-y)}{(1-y)}\ dydx$$

can not be solved. We have to change the order of integration:

$$\int_0^1 \int_y^1 \frac{\sin(1-y)}{(1-y)}\ dxdy = \int_0^1 \sin(1-y)\ dy = \cos(1-y)|_0^1 = 1 - \cos(1)\ .$$

The final result is $\boxed{1 - \cos(1)}$.

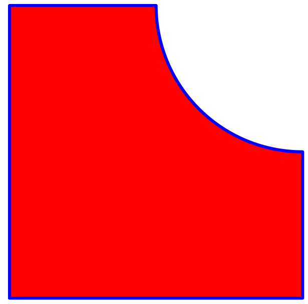
Problem 11) (10 points)

Let $\vec{F} = \langle y, 2x + \tan(\tan(y)) \rangle$ be a vector field in the plane and let C be the boundary of the region

$$G = \{0 \le x \le 2, 0 \le y \le 2, (x-2)^2 + (y-2)^2 \ge 1\ \}$$

oriented counter clock-wise. Compute the line integral

$$\int_C \vec{F}\ \cdot d\vec{r}\ .$$



Solution:
The curl of $\vec{F} = \langle P, Q \rangle$ is constant $Q_x - P_y = 2 - 1 = 1$. The line integral by **Greens theorem**

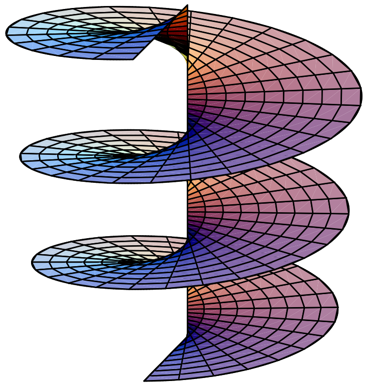
$$\int_C \vec{F}\ d\vec{r} = \int\int_G \vec{F}\ dA$$

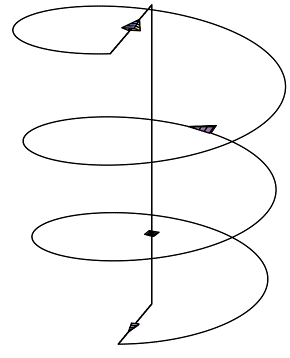
equal to the area of the region, which is the area of the square minus the area of the quarter disc: $\boxed{4 - \pi/4}$.

Problem 12) (10 points)

Let S be the surface of a turbine blade parameterized by $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), t \rangle$ for $t \in [0, 6\pi]$ and $s \in [0,1]$. Let $\vec{F} = \mathrm{curl}(\vec{G})$ denote the velocity field of the water velocity, where $\vec{G}(x,y,z) = \langle -y + (x^2+y^2-1), x + (x^2+y^2-1), (x^2+y^2-1) \rangle$. Compute the power of the turbine which is given by the flux of $\vec{F} = \mathrm{curl}(\vec{G})$ through S.

Hint. The boundary C of the surface S consists of 4 paths:
$\vec{r_1}(t) = \langle \cos(t), \sin(t), t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_2}(s) = \langle 1-s, 0, 6\pi \rangle$, $s \in [0,1]$.
$\vec{r_3}(t) = \langle 0, 0, 6\pi - t \rangle$, $t \in [0, 6\pi]$.
$\vec{r_4}(s) = \langle s, 0, 0 \rangle$, $s \in [0,1]$.





Solution:
We use **Stokes theorem**

$$\int\int_S \text{curl}(\vec{G})\ dS = \int_C \vec{G}\ d\vec{r}\ .$$

This allows us to compute the line integral along the boundary instead of the flux. This line integral consists of 4 paths. The line integrals along $\vec{r}_2$ and $\vec{r}_4$ cancel, one is $2/3$, the other $-2/3$. On $\vec{r}_1$, the vector field is $\vec{G}(x,y,z) = \langle -y, x, 0\rangle$ and $\vec{G}(\vec{r}(t)) = \langle -\sin(t), \cos(t), 0\rangle$ is parallel to $\vec{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle$. Computing the line integral gives $\int_0^{6\pi} 1\ dt = 6\pi$. On $\vec{r}_3$, both x and y are zero so that the vector field is $\langle -1,-1,-1\rangle$ and the line integral gives $\int_0^{6\pi}\langle -1,-1,-1\rangle \cdot \langle 0,0,-1\rangle\ dt = 6\pi$. The sum of all four line integrals is $\boxed{12\pi}$.

Problem 13) (10 points)

Let $\vec{F}(x,y,z) = \langle z^2, -z^5 + z\sin(e^{\sin(x)}), (x^2+y^2)\rangle$. Let S denote the part of the graph $z = 9-x^2-y^2$ lying above the xy-plane oriented so that the normal vector points upwards. Find the flux of $\vec{F}$ through the surface S.

Hint. You might also want to look at the surface $D = \{x^2+y^2 \leq 9, z = 0\ \}$ lying in the xy-plane.

Solution:
The vector field is incompressible. By the **divergence theorem**, if D is oriented upwards, we have

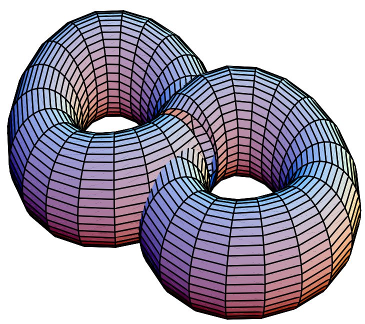
$$\int\int_S \vec{F}\ dS - \int\int_D \vec{F}\ dS = 0\ .$$

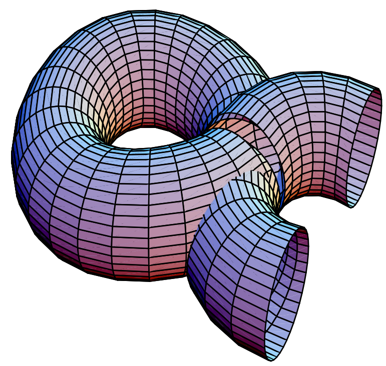
The flux through S is therefore the same as the flux through D if both D and S are oriented upwards. The flux through D can be computed easily because $\vec{F} = \langle 0,0,x^2+y^2\rangle$ there. That integral is best evaluated in polar coordinates $\int_0^3\int_0^{2\pi} r^3\ d\theta dr = 3^4 2\pi/4 = \boxed{81\pi/2}$.

Problem 14) (10 points)

We are given two vector fields $\vec{F}(x,y,z) = \langle x,y,z \rangle$ and $\vec{G}(x,y,z) = \langle y,z,x \rangle$. We are also given a brezel surface S bounding a solid of volume 3 and a surface U which is part of S and has been left after cutting away a piece of the brezel. The surface U together with two discs D_1 and D_2 form a closed surface bounding a "cut brezel". The first surface S does not have a boundary and is oriented so that the normal vector points outwards. The second surface U has the same orientation than S and as a boundary the union C of two curves C_1 and C_2 which are oriented so that U is "to the left", when you look into the direction of the velocity vector and if the normal vector to U points "up".

a) (2 points) Compute $\int\int_S \vec{F}\; d\vec{S}$.
b) (2 points) Find $\int\int_S \vec{G}\; d\vec{S}$.
c) (2 points) Verify that the vector field $\vec{A} = \langle -xy, -yz, -zx \rangle$ is a **vector potential** of $\vec{G}$ meaning that $\vec{G} = \text{curl}(\vec{A})$.
d) (2 points) Express $\int_C \vec{A} \cdot\; d\vec{r}$ as a flux integral through U.
e) (2 points) Compute $\int_C \vec{F} \cdot\; d\vec{r}$.





Solution:
a) Use the divergence theorem. div$\vec{F} = 3$, the integral is 3 times the volume of the solid, which is $\boxed{9}$.
b) Again by thd divergence theorem, now using div$\vec{F} = 0$, the flux integral is $\boxed{0}$.
c) This is a computation: $\nabla \times \vec{A} = \vec{G}$: the curl is the determinant of

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -xy & -yz & -zx \end{bmatrix} = y\vec{i} + z\vec{j} + x\vec{k} = \langle y, z, x \rangle \ .$$

d) By Stokes theorem, the integral is the flux of $\vec{G}$ through U. The orientation is the orientation which is required by Stokes theorem. The answer is $\boxed{\int\int_U \vec{G} \cdot d\vec{S}}$.
e) There are two ways to see this: By Stokes theorem, the integral is the flux of curl(F) through U. But because $\vec{F}$ is a gradient field, its curl is 0.
One can also see this by the fundamental theorem of line integrals. Because $\vec{F} = \nabla f$ for $f(x, y, z) = (x^2 + y^2 + z^2)/2$, and the curves are closed curves, the integral is $\boxed{0}$.