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TTH 11:30 Jameel Al-Aidroos

- Printing your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) [T] [F] If $f(x, y, z)$ is a function then the line integral of curl(∇f)) around any closed circle is zero.

Solution:
The line integral of the gradient would be zero, but not of the curl. So, one could think, it is false, But it is true because $curl(grad(f)) = 0$.

2) [T] [F] If E is the solid half-sphere $x^2 + y^2 + z^2 \leq 1, z < 0$ then $\iiint_E x^4 \; dxdydz$ is positive.

Solution:
It is an integral of a positive function.

3) [T] [F] If the vector field $\vec{F}$ is incompressible and S and R are surfaces with the same boundary C and orientation then $\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot d\vec{S}$.

Solution:
This follows from the divergence theorem.

4) [T] [F] $\vec{r}(u, v) = \langle \cos(u), \sin(u), 0 \rangle + v \langle -\sin(u), \cos(u), 1 \rangle$ parametrizes a surface in the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution:
Plug in x, y, z into the equation.

5) [T] [F] The equation $\text{div}(\text{grad} f) = |\text{grad} f|^2$ is an example of a partial differential equation for the unknown function $f(x, y, z)$.

Solution:
If we write it out, it is.

6) [T] [F] The vector $(\vec{i} \times \vec{j}) \times \vec{i}$ is the zero vector.

Solution:
$\vec{i} \times \vec{j}$ is equal to $\vec{k}$.

7) T F The length of the gradient $|\nabla f|$ is minimal at a local minimum of $f(x, y)$ if a local minimum of f exists.

Solution:
It is zero at a minimum.

8) T F The length of the gradient $|\nabla f|$ is maximal at a local maximum of $f(x, y)$ if a local maximum of f exists.

Solution:
It is zero at a maximum.

9) T F If $\vec{r}(t)$ parametrizes the curve obtained by intersecting $y = 0$ with $x^2 + y^2 + z^2 = 1$, then the bi-normal vector $\vec{B}$ is tangent to the surface.

Solution:
The normal vector points into the sphere towards the center. The binormal vector is both orthogonal to the tangent and normal vector. It is tangent to the sphere.

10) T F If the line integral of $\vec{F}$ along the closed loop $x^2 + y^2 = 1, z = 0$ is zero then the vector field is conservative.

Solution:
It needs to be with respect to all paths.

11) T F The vectors $\vec{v} = \langle 1, 0, 0 \rangle$ and $\vec{w} = \langle 1, 1, 0 \rangle$ have the property that $\vec{v} \cdot \vec{w} = |\vec{v} \times \vec{w}|$.

Solution:
Compute both, and it is almost true. In two dimensions, the cross product is also a scalar. But the sign is off.

12) T F The surface area of a sphere depends on the orientation of the sphere. It is positive if the normal vector points outwards and changes sign if the orientation is changed.

Solution:
It is the flux which depends on the orientation.

13) T F If $f(x,y,z) = (\text{div}(\vec{F})(x,y,z))$ has a maximum at $(0,0,0)$, then $\text{grad}(\text{div}(\vec{F}))(0,0,0) = \langle 0,0,0 \rangle$.

Solution:
Yes, f is a function.

14) T F The curl of a conservative vector field is zero.

Solution:
This is essentially Clairot.

15) T F The flux of the curl of $\vec{F}$ through a disc $x^2+y^2 \leq 1, z=0$ is always zero.

Solution:
By Stokes theorem, it would be zero through a closed surface. But the disc is not closed.

16) T F If the integral $\int\int\int_G \text{div}(\vec{F}(x,y,z))\, dxdydz$ is zero for the ball $G = \{x^2+y^2+z^2 \leq 1\}$, then the divergence is zero at $(0,0,0)$.

Solution:
Take $F = \langle x^3, 0, 0 \rangle$

17) T F The value $\sqrt{101 \cdot 10002}$ can by linear approximation be estimated as $1000 + 5 \cdot 1 + (1/20) \cdot 2$.

Solution:
The function $f(x,y) = \sqrt{101 \cdot 10002}$ is linearized by $L(x,y) = 1000 + 5(x-100) + (1/2)(y-10000)$.

18) T F If $\vec{F} = \text{curl}(\vec{G})$ and $\text{div}(\vec{F}) = 0$ everywhere in space, then $\text{div}(\vec{G}) = 0$ everywhere in space.

Solution:
$\text{div}(\vec{F}) = 0$ is always true in that case.

19) | T | F | It is possible that $\vec{v} \cdot \vec{w} > 0$ and $\vec{v} \times \vec{w} = \vec{0}$.

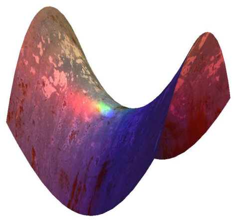
Solution:
Take $\vec{v} = \vec{w}$.

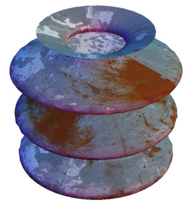
20) | T | F | The directional derivative $D_{\vec{v}}(f)$ is defined as $\nabla f \times \vec{v}$.

Solution:
It is the dot product not the cross product.

Problem 2) (10 points) No justifications are necessary.

a) (3 points) The following surfaces are given either as a parametrization or implicitly in some coordinate system (Cartesian, cylindrical or spherical). Each surface matches exactly one definition.


A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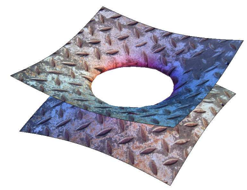
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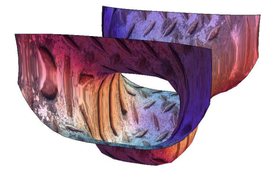
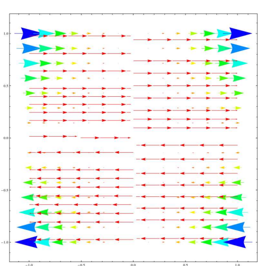

D

Enter A-D here	Function or parametrization
	$r = 3 + 2\sin(3z)$
	$\vec{r}(u,v) = \langle u, v, u^2 - v^2\rangle$
	$x^4 - zy^4 + z^4 = 1$
	$r^2 - 8z^2 = 1$

b) (3 points) The pictures display flow lines of vector fields in two dimensions. Match them.

Field	Enter 1-4
$\vec{F}(x,y) = \langle 0, x^2y\rangle$	
$\vec{F}(x,y) = \langle x^2y, 0\rangle$	
$\vec{F}(x,y) = \langle -y-x, x\rangle$	
$\vec{F}(x,y) = \langle -y, x\rangle$	

1


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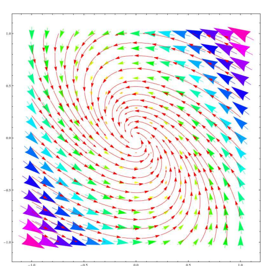

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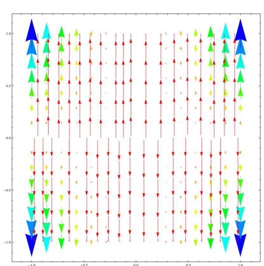

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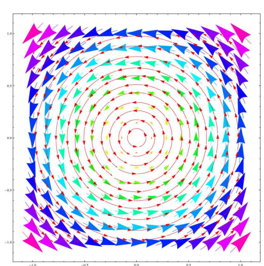


c) (2 points) Match the following partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-C	1-3
Laplace		
wave		
heat		

A	$u(t,x) = t + t^2 - x^2$
B	$u(t,x) = t + t^2 + x^2$
C	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_{xx}(t,x)$
2	$u_{tt}(t,x) = -u_{xx}(t,x)$
3	$u_{tt}(t,x) = u_{xx}(t,x)$

d) (2 points) Two of the six expressions are **not** independent of the parametrization. Check them.

Velocity $\vec{r}\,'(t)$		Surface area $\int\int |\vec{r}_u \times \vec{r}_v| dudv$		Line integral $\int_a^b \vec{F} \cdot d\vec{r}$	
Arc length $\int |\vec{r}\,'(t)|\, dt$		Flux integral $\int\int_R \vec{F} \cdot d\vec{S}$		Normal vector $\vec{r}_u \times \vec{r}_v$	

Solution:
a) BADC
b) 3124
c) ABC, 231
d) Velocity and Normal vector. The Surface area and line integral does not depend on the parametrization for fixed orientation.

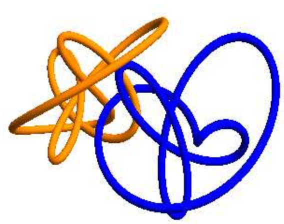
Problem 3) (10 points)

a) (4 points) The following objects are defined in three dimensional space.
Fill in either "surface", "curve", or "vector field" in each case.

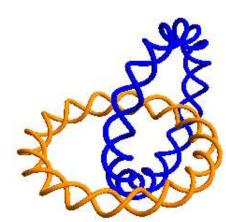
formula	surface, curve or field?
$x + y = 1$	
$x + y = 1, x - y = 5$	
$\vec{F}(x, y, z) = \langle x, x + y, x - y \rangle$	
$\vec{r}(x, y) = \langle x, y, x - y \rangle$	
$\vec{r}(x) = \langle x, x, x^2 - x \rangle$	

b) (2 points) Two closed curves $\vec{r}_1(t), \vec{r}_2(t)$ form a **link**. In our case, the curve $\vec{r}_2(t)$ is a copy of the other moved and turned around. Match three of them. Links and knots are relevant in biology: DNA strands can forms links or knots which need disentanglement.

$\vec{r}_1(t)$	Enter A,B,C,D
$\langle 7\cos(t), 7\sin(t), 7\cos(t) \rangle$	
$\langle (7 + \sin(17t))\cos(t), (7 + \cos(17t))\sin(t), \sin(17t) \rangle$	
$\langle \cos(2t) + \sin(4t), \cos(4t) + \cos(3t), \cos(2t) + \sin(3t) \rangle$	



A



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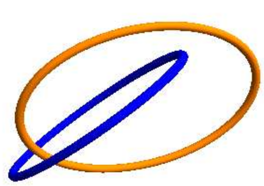

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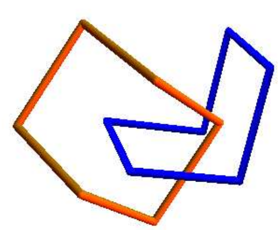


D

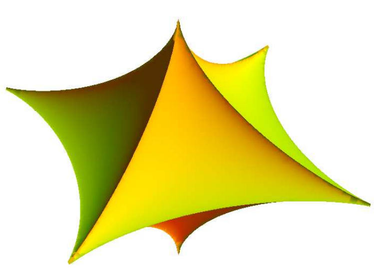
c) (4 points) Which of the following expressions are defined if $\vec{F}(x, y, z)$ is a vector field and $f(x, y, z)$ a scalar field in space. Is the result a scalar or vector field?

Formula	Defined	Not defined	Scalar	Vector
curl(grad(div($\vec{F}$)))				
curl(div(grad(f)))				
grad(div(curl($\vec{F}$)))				
grad(curl(div($\vec{F}$)))				
div(curl(grad(f)))				
div(grad(curl(f)))				

Solution:
a) surface, curve, field, surface, curve
b) CBA

Formula	Defined	Not defined	Scalar	Vector
curl(grad(div($\vec{F}$)))	x			x
curl(div(grad(f)))		x		
grad(div(curl($\vec{F}$)))	x			x
grad(curl(div($\vec{F}$)))		x		
div(curl(grad(f)))	x		x	
div(grad(curl(f)))		x		

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the surface S given by

$$3x^{2/3} + 3y^{2/3} + 6z^{2/3} = 12$$

at the point $(1, 1, 1)$.

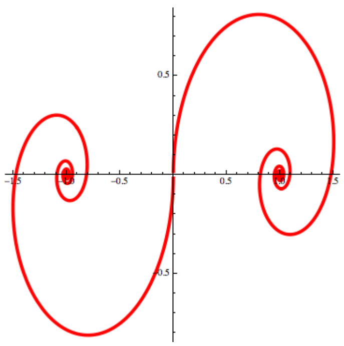
b) (5 points) When S is intersected with the plane $y = 1$, we get the curve

$$3x^{2/3} + 6z^{2/3} = 9 \ .$$

Find the tangent line of the form $ax + bz = d$ for the tangent line at $(x, z) = (1, 1)$.

Solution:
a) $\nabla f(1,1,1) = \langle 2,2,4 \rangle$ so that $2x + 2y + 4z = 8$ is the solution.
b) $\nabla f(1,1) = \langle 2,4 \rangle$ so that $x + 2z = 3$ is the solution.

Problem 5) (10 points)



Find the line integral for the vector field

$$\vec{F}(x,y) = \langle x^6 + y + 3x^2y^3, y^7 + x + 3x^3y^2 \rangle$$

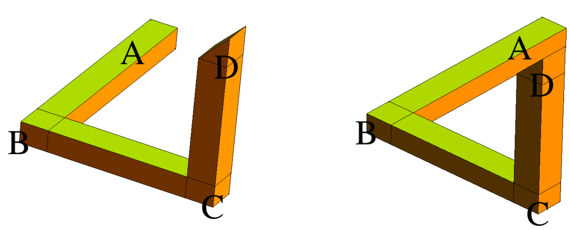
along the **ornamental curve**

$$\vec{r}(t) = \langle \frac{t}{|t|}(1 - \frac{1}{1+t^2}\cos(4t)), \frac{1}{1+t^2}\sin(4t) \rangle$$

from $t = -\infty$ to $t = \infty$. This curve connects the point $(-1,0)$ with $(1,0)$ along an **infinite epic journey**.

Solution:
The vector field is a gradient field with $f(x,y) = x^7/7 + xy + y^8/8 + x^3y^3$. By the fundamental theorem of line integrals, we get $f(r(b)) - f(r(a)) = (1/7) - (-1/7) = 2/7$.

Problem 6) (10 points)



The **Penrose tribar** is a path in space connecting the points $A = (1,0,0), B = (0,0,0), C = (0,0,1), D = (0,1,1)$. While the distance between A and D is positive, we see an impossible triangle when the line of sight goes through A and D.

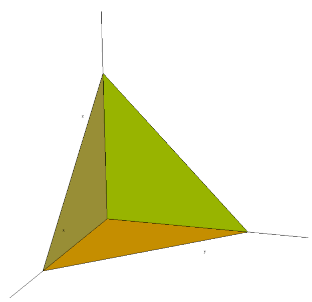
a) (2 points) Find the distance between the points A and D.

b) (4 points) Parametrize the line connecting A and D.

c) (4 points) Find the distance between the lines AB and CD.

Solution:
a) $\sqrt{3}$ by Pythagoras or distance formula.
b) $\vec{r}(t) = \langle 1-t, t, t \rangle$
c) Use the distance formula using $\vec{v} = \vec{AB} = \langle -1, 0, 0 \rangle$ and $\vec{w} = \vec{CD} = \langle 0, 1, 0 \rangle$ and two points $P = (0, 0, 0)$ and $Q = (0, 0, 1)$ to get $d = 1$.

Problem 7) (10 points)



You invent a **3D printing process** in which materials of variable density can be printed. To try this out, we take a tetrahedral region E:

$$x + y + z \leq 1; x \geq 0, y \geq 0, z \geq 0$$

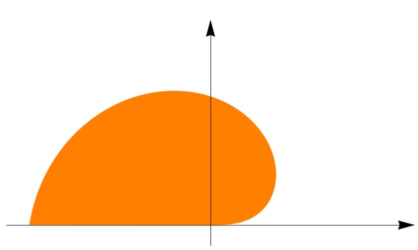
which has the density $f(x, y, z) = 24x$. Find the total mass

$$\int\int\int_E f(x, y, z)\ dxdydz\ .$$

Solution:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 24x\ dzdydx = 1$$

Problem 8) (10 points)



When we integrate the function $f(x, y) = 2y/(\sqrt{x^2 + y^2}\arctan(y/x))$ over the **snail region** $r^2 \leq \theta \leq \pi$, we are led in polar coordinates to the integral

$$\int_0^{\sqrt{\pi}} \int_{r^2}^{\pi} \frac{2r\sin(\theta)}{\theta}\ d\theta\ dr$$

Evaluate this integral.

Solution:

$$\int_0^{\sqrt{\pi}} \int_{r^2}^{\pi} 2r \sin(\theta)/\theta \; d\theta \; dr$$

Changing the order of integration gives

$$\int_0^{\pi} \int_0^{\sqrt{\theta}} 2r \sin(\theta)/\theta \; dr \; d\theta = 2$$

Problem 9) (10 points)

Old Mc Donald wants to build a farm on a location, where the ground is as even as possible. Let $g(x, y) = y^2 + xy + x$ be the height of the ground. Find the point (x, y), where the **steepness**

$$f(x, y) = |\nabla g|^2$$

is minimal. Classify all critical points of f.

Solution:
The function under consideration is $f(x, y) = (1 + y)^2 + (x + 2y)^2$. There is one critical point $(2, -1)$ which has discriminant 4 and $f_{xx} = 2$. It is a minimum.

Problem 10) (10 points)

We build a **bike** which has as a frame a triangle of base length x and height y and a wheel which has radius y. Using Lagrange, find the bike which has maximal

$$f(x, y) = xy + 4\pi y^2$$

(which is twice the area) under the constraint

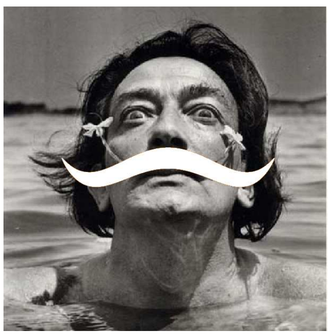
$$g(x, y) = x + 10\pi y = 3 \; .$$

Solution:
The Lagrange equations are

$$\begin{aligned} y &= \lambda \\ x + 8\pi y &= \lambda(10\pi) \\ x + 10\pi y &= 3 \end{aligned}$$

It has only one solution $(x, y) = (1/2, 1/(4\pi))$, which is a maximum.

Problem 11) (10 points)



Compute the area of the **moustache region** which is enclosed by the curve

$$\vec{r}(t) = \langle 5\cos(t), \sin(t) + \cos(4t) \rangle$$

with $0 \leq t \leq 2\pi$.

Hint. You can use without justification that integrating an odd 2π periodic function from 0 to 2π is zero.

Solution:
Use the vector field $\vec{F}(x, y) = \langle 0, x \rangle$ and use Greens theorem. We have

$$\int_0^{2\pi} \langle 0, 5\cos(t) \rangle \cdot \langle -5\sin(t), \cos(t) - 4\sin(4t) \rangle \, dt = 5\pi$$

We have used that $\int_0^{2\pi} \cos^2(t) \, dt = \pi$ (double angle formula) and the hint. The hint had not be justified and it is obvious. If you look at an odd function like $\sin(4x)\cos(x)$ [odd f means $f(-x) = -f(x)$], then the contribution on $[0, \pi]$ is canceled by the contribution on $[-\pi, 0]$. During the exam a proctor was asked what it means for a function to be "strange". An odd question! The proctor integrated it to zero.

Problem 12) (10 points)

We enjoy the pre-holiday season in a local Harvard square coffee shop, where **coffee aroma** diffuses in the air. Find the flux of the air velocity field

$$\vec{F}(x,y,z) = \langle y^2, x^2, z^2 \rangle$$

leaving a coffee box

$$E : x^2 + y^2 \leq 1, x^2 + y^2 + z^2 \leq 4 \ .$$

Solution:
Use the divergence theorem and integrate

$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 2zr \ dzdrd\theta \ = 0 \ .$$

In this 21a coffee box, the total coffee aroma stays. Aroma leaves on the upper part and enteres in the lower part.

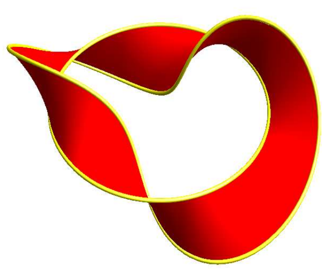
Problem 13) (10 points)

Find the line integral of

$$\vec{F}(x,y,z) = \langle -y, x, e^{\sin z} \rangle$$

along the positively oriented boundary of the **ribbon** $\vec{r}(u,v)$ parametrized on $0 \leq u \leq 4\pi$ and $0 \leq v \leq 1/2$ with

$$\vec{r}(u,v) = \langle (1+v\cos(2u))\cos(u), (1+v\cos(2u))\sin(u), v\sin(2u) \rangle$$



for which a **good fairy** gives you the normal vector

$$\begin{aligned} \vec{r}_u \times \vec{r}_v \quad = \quad & \langle -\sin(u)(v\cos(4u) + 2(v+1)\cos(2u) - 3v + 2)/2 \ , \\ & \cos(u)(v\cos(4u) - 2(v-1)\cos(2u) - 3v - 2)/2 \ , \\ & -\cos(2u)(v\cos(2u) + 1) \rangle \ . \end{aligned}$$

Solution:
We use Stokes, but this time, we compute the flux. We get

$$\int_0^{4\pi}\int_0^{1/2} \langle 0,0,2\rangle \cdot \langle \dots, \dots, -\cos(2u)(v\cos(2u)+1)\rangle \, dvdu$$

which simplifies to $-\pi/2$.
Remarks
1) One student was intimidated by the complexity of the normal vector and questioned the name "good fairy". While writing the exam, we actually debated for a while whether to give the normal vector. We were not persuaded by a "good fairy" but the prospect of having to grade 1000 pages of futile and probably wrong computations of $\vec{r}_u \times \vec{r}_v$. The above expressions are already simplified.
2) The ribbon is a variant of a Möbius strip. But unlike the famous Moebius strip, it is orientable. The surface has one side. We were actually surprised to see that the flux of the vector field $\langle 0,0,2\rangle$ is not zero because one would expect by symmetry to get zero. Actually, the above parametrization goes over the ribbon twice. We could go from 0 to 2π too.

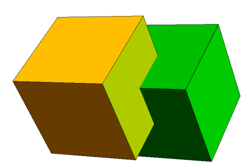
Problem 14) (10 points)

A computer can determine the volume of a solid enclosed by a triangulated surface by computing the fluxe of the vector field $\vec{F} = \langle 0,0,z\rangle$ through each triangle and adding them all up. Lets go backwards and compute the flux of this vector field $\vec{F} = \langle 0,0,z\rangle$ through the surface S which bounds a solid called "abstract cow" (this is avant-garde **"neo-cubism"** style)

$$\{0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\} \ \cup$$

$$\{1 \le x \le 3, 1 \le y \le 3, 1 \le z \le 3\} \ ,$$

where $\cup$ is the union and the surface is oriented outwards.



Ceci n'est pas une pipe
Ceci c'est une vache

Solution:
The divergence is 1. The volume of the cubistic cow is $2^3 + 2^3 - 1 = 15$. The divergence theorem gives the answer 15.