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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
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11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

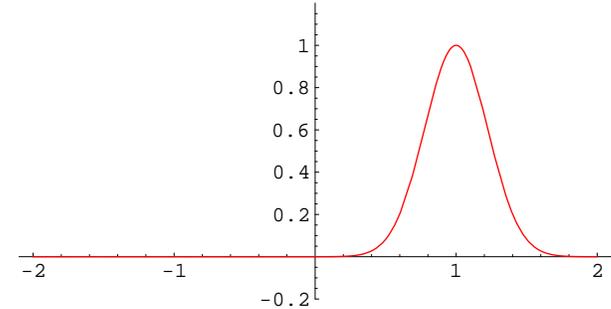
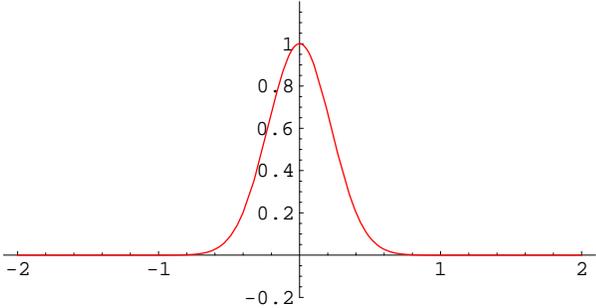
- 1)  T  F The distance from  $(1, 2, -1)$  to  $(3, -2, 1)$  is  $(-2, 4, -2)$ .
- 2)  T  F The plane  $y = 3$  is perpendicular to the  $xz$  plane.
- 3)  T  F All functions  $u(x, y)$  that obey  $u_x = u$  at all points obey  $u_y = 0$  at all points.
- 4)  T  F The best linear approximation at  $(1, 1, 1)$  to the function  $f(x, y, z) = x^3 + y^3 + z^3$  is the function  $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$
- 5)  T  F If  $f(x, y)$  is any function of two variables, then  $\int_0^1 \left( \int_x^1 f(x, y) dy \right) dx = \int_0^1 \left( \int_y^1 f(x, y) dx \right) dy$ .
- 6)  T  F Let  $C = \{(x, y) \mid x^2 + y^2 = 1\}$  be the unit circle in the plane and  $\vec{F}(x, y)$  a vector field satisfying  $|\vec{F}| \leq 1$ . Then  $-2\pi \leq \int_C \vec{F} \cdot d\vec{r} \leq 2\pi$ .
- 7)  T  F Let  $\vec{a}$  and  $\vec{b}$  be two nonzero vectors. Then the vectors  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  always point in different directions.
- 8)  T  F If all the second-order partial derivatives of  $f(x, y)$  vanish at  $(x_0, y_0)$  then  $(x_0, y_0)$  is a critical point of  $f$ .
- 9)  T  F If  $\vec{a}, \vec{b}$  are vectors, then  $|\vec{a} \times \vec{b}|$  is the area of the parallelogram determined by  $\vec{a}$  and  $\vec{b}$ .
- 10)  T  F The distance between two points  $A, B$  in space is the length of the curve  $\vec{r}(t) = A + t(B - A)$ ,  $t \in [0, 1]$ .
- 11)  T  F The function  $f(x, y) = xy$  has no critical point.
- 12)  T  F The length of a curve does not depend on the chosen parameterization.
- 13)  T  F There exists a non-zero function  $f(x, y, z)$  and non-zero vector field  $\vec{F}(x, y, z)$  so that  $\vec{F} = \text{grad}(f)$  and  $f = \text{div}\vec{F}$ .
- 14)  T  F For any numbers  $a, b$  satisfying  $|a| \neq |b|$ , the vector  $\langle a - b, a + b \rangle$  is perpendicular to  $\langle a + b, b - a \rangle$ .
- 15)  T  F The line integral of  $\vec{F}(x, y) = \langle -y, x \rangle$  along the counterclockwise oriented boundary of a region  $R$  is twice the area of  $R$ .
- 16)  T  F There is no surface for which both the parabola and the hyperbola appear as traces.
- 17)  T  F If  $(u, v) \mapsto \vec{r}(u, v)$  is a parameterization for a surface, then  $\vec{r}_u(u, v) + \vec{r}_v(u, v)$  is a vector which lies in the tangent plane to the surface.
- 18)  T  F When using spherical coordinates in a triple integral, one needs to include the volume element  $dV = \rho^2 \cos(\phi) d\rho d\phi d\theta$ .
- 19)  T  F A surface in space for which all normal vectors are parallel to each other must be part of a plane.
- 20)  T  F A vector field  $\vec{F} = \langle P(x, y), Q(x, y) \rangle$  is conservative in the plane if and only if  $P_y(x, y) = Q_x(x, y)$  for all points  $(x, y)$ .

Problem 2) (10 points)

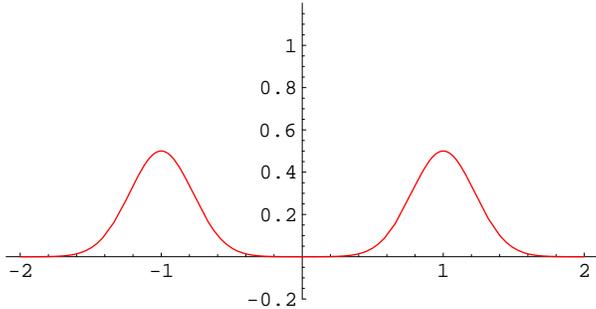
2 a) (5 points) Fill in names of the mathematicians: Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

Formula	Name of the theorem
$\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x, y) = f_{yx}(x, y)$	
$\int_C \vec{F} \cdot d\vec{r} = \int \int_R \text{curl}(\vec{F}) \, dx dy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int \int_S \vec{F} \cdot dS = \int \int \int_E \text{div}(\vec{F}) \, dV$	
$\int_a^b \int_c^d f(x, y) \, dx dy = \int_c^d \int_a^b f(x, y) \, dy dx$	

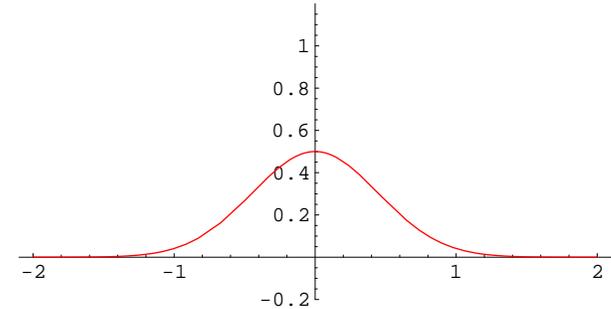
2 b) (5 points) We have a function  $u(t, x)$  which is a solution to a partial differential equation. In all cases, we have  $u(0, x) = e^{-x^2}$ . The picture to the right shows this function  $u(0, x)$ . Which partial differential equation is involved, when you see the function  $u(1, x)$  as a graph?



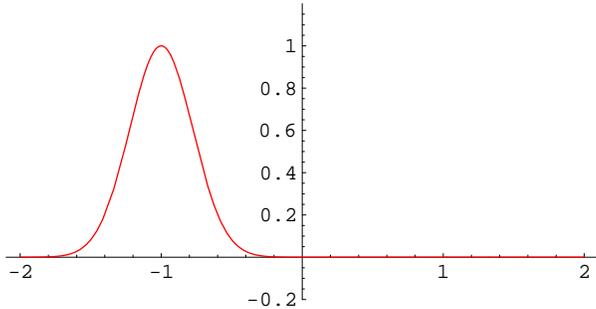
I



II



III



IV

Enter I,II,III,IV here	Equation
	$u_t(x, t) = u_x(x, t)$
	$u_t(x, t) = u_{xx}(x, t)$
	$u_{tt}(x, t) = u_{xx}(x, t)$
	$u_t(x, t) = -u_x(x, t)$

Problem 3) (10 points)

- a) Find an equation for the plane  $\Sigma$  passing through the points  $P = (1, 0, 1)$ ,  $Q = (2, 1, 3)$  and  $R = (0, 1, 5)$ .
- b) Find the distance from the origin  $O = (0, 0, 0)$  to  $\Sigma$ .
- c) Find the distance from the point  $P$  to the line through  $Q, R$ .
- d) Find the volume of the parallelepiped with vertices  $O, P, Q, R$ .

Problem 4) (10 points)

The equation  $f(x, y, z) = e^{xyz} + z = 1 + e$  implicitly defines  $z$  as a function  $z = g(x, y)$  of  $x$  and  $y$ .

- a) Find formulas (in terms of  $x, y$  and  $z$ ) for  $g_x(x, y)$  and  $g_y(x, y)$ .
- b) Estimate  $g(1.01, 0.99)$  using linear approximation.

Problem 5) (10 points)

Find the surface area of the surface  $S$  parametrized by  $\vec{r}(u, v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$  for  $(u, v)$  in the disc  $D = \{u^2 + v^2 \leq 1\}$ .

Problem 6) (10 points)

Find the local and global extrema of the function  $f(x, y)$  which is the curl of  $\vec{F}(x, y) = \langle -y^4/12 + y^3/6 - y, x^4/12 - x^3/6 \rangle$  on the disc  $\{x^2 + y^2 \leq 4\}$ .

- a) Classify every critical point inside the disc  $x^2 + y^2 < 4$ .
- b) Find the extrema on the boundary  $\{x^2 + y^2 = 4\}$  using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of  $D$ .

Problem 7) (10 points)

a) Given two nonzero vectors  $\vec{u} = \langle a, b, c \rangle$  and  $\vec{v} = \langle d, e, f \rangle$  in space, write down a formula for the cosine of the angle between them. Find a nonzero vector  $\vec{v}$  that is perpendicular to  $\vec{u} = \langle 3, 2, 1 \rangle$ . Describe geometrically the set of all  $\vec{v}$ , including zero, that are perpendicular to this vector  $\vec{u}$ .

b) Consider a function  $f$  of three variables. Explain with a picture and a sentence what it means geometrically that  $\nabla f(P)$  is perpendicular to the level set of  $f$  through  $P$ .

c) Assume the gradient of  $f$  at  $P$  is nonzero. Write a few sentences that would convince a skeptic that  $\nabla f(P)$  is perpendicular to the level set of  $f$  at the point  $P$ .

d) Assume the level set of  $f$  is the graph of a function  $g(x, y)$ . Explain the relation between the gradient of  $g$  and the gradient of  $f$ . Especially, how do you relate the orthogonality of  $\nabla f$  to the level set of  $f$  with the orthogonality of  $\nabla g$  to the level set of  $g$ ?

Problem 8) (10 points)

Let  $R$  be the region inside the circle  $x^2 + y^2 = 4$  and above the line  $y = \sqrt{3}$ . Evaluate

$$\iint_R \frac{y}{x^2 + y^2} dA.$$

Problem 9) (10 points)

A region  $W$  in  $\mathbf{R}^3$  is given by the relations

$$\begin{aligned}x^2 + y^2 &\leq z^2 \leq 3(x^2 + y^2) \\ 1 &\leq x^2 + y^2 + z^2 \leq 4 \\ x &\geq 0\end{aligned}$$

1. Sketch the region  $W$ .
2. Find the volume of the region  $W$ .

Problem 10) (10 points)

Consider the vector field

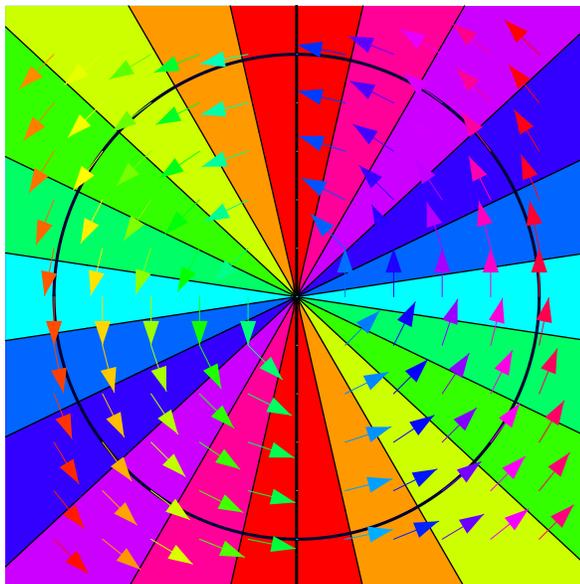
$$\vec{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

defined everywhere in the plane  $\mathbf{R}^2$  except at the origin.

a) Let  $C$  be any closed curve which bounds a region  $D$ . Assume that  $(0, 0)$  is not contained in  $D$  and does not lie on  $C$ . Explain why

$$\int_C \vec{F} \cdot d\vec{r} = 0 .$$

b) Let  $C$  be the unit circle oriented counterclockwise. What is  $\int_C \vec{F} \cdot d\vec{r}$ ? Explain why your answer shows that there is no function  $f$  for which  $\vec{F}(x, y) = \nabla f(x, y)$  everywhere.



Problem 11) (10 points)

First use rectangular, then cylindrical and finally spherical coordinates to integrate the function  $f(x, y, z) = xyz$  over the solid in space described by the inequalities  $0 \leq z \leq \sqrt{1 - x^2 - y^2}$ ,  $x^2 + y^2 \leq 1$ ,  $x - y \geq 0$ ,  $y \geq 0$ .

Problem 12) (10 points)

Let  $\vec{F}(x, y)$  be a vector field in the plane given by the formula

$$\vec{F}(x, y) = \langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4 + 1}} \rangle.$$

If  $C$  is the path which goes from from  $(-1, 0)$  to  $(1, 0)$  along the semi circle  $x^2 + y^2 = 1$ ,  $y \geq 0$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$ .

Problem 13) (10 points)

In appropriate units, the charge density  $\sigma(x, y, z)$  in a region in space is given by  $\sigma = \nabla \cdot \vec{E} = \text{div}(\vec{E})$ , where  $\vec{E}$  is the electric field. Consider the cube of side lengths 1 given by  $0 \leq x, y, z \leq 1$ . What is the total charge in this cube if

$$\vec{E} = \langle x(1-x) \log(1+xyz), y(1-y) \tan(x^3 + y^3 + z^3), z(1-z)e^{\sqrt{x+y}} \rangle.$$

(The total charge is the integral of the charge density over the cube.)

Problem 14) (10 points)

a) By calculating the integral  $\int \int_S \vec{F} \cdot d\vec{S}$  directly, find the flux of the vector field  $\vec{F}(x, y, z) = \langle 0, 0, x+z \rangle$  through the sphere  $x^2 + y^2 + z^2 = 9$ , where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field  $\vec{F}(x, y, z) = \langle 0, 0, x+z \rangle$  through the sphere  $x^2 + y^2 + z^2 = 9$  using the divergence theorem.

c) Explain in words without invoking any integral theorem, why the flux integral of the vector field  $\vec{F}(x, y, z) = \langle 0, 0, x+z \rangle$  through any sphere with positive radius centered at  $(0, 0, 0)$  is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.