

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Yu-Wen Hsu
MWF 10 Yong-Suk Moon
MWF 11 Rosalie Belanger-Rioux
MWF 11 Gijs Heuts
MWF 11 Siu-Cheong Lau
MWF 12 Erick Knight
MWF 12 Kate Penner
TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- 1) ☐ T ☒ F The directional derivative $D_{\vec{v}}f$ is a vector perpendicular to $\vec{v}$.

Solution:

The directional derivative is a scalar, not a vector.

- 2) ☒ T ☐ F Using linearization of $f(x, y) = xy$ we can estimate $f(0.9, 1.2) \sim 1 - 0.1 + 0.2 = 1.1$.

Solution:

$$L(x, y) = 1 - 1 \cdot 0.1 + 1 \cdot 0.2.$$

- 3) ☒ T ☐ F Given a curve $\vec{r}(t)$ on a surface $g(x, y, z) = 1$, then $\frac{d}{dt}g(\vec{r}(t)) = 0$.

Solution:

This fact is used in the proof that level surfaces are perpendicular to gradients.

- 4) ☐ T ☒ F Given a function $f(x, y)$ such that $\nabla f(0, 0) = \langle 2, -1 \rangle$. Then $D_{\langle 0, -1 \rangle} f(0, 0) = 0$.

Solution:

The directional derivative in the direction $\langle 0, -1 \rangle$ is equal to -1 which is nonzero.

- 5) ☐ T ☒ F $\vec{r}(u, v) = \langle u \cos(v), u \sin(v), v \rangle$ is a surface of revolution.

Solution:

The parametrization given is a helicoid and not rotationally symmetric. It resembles the parametrization $\vec{r}(u, v) = \langle u \cos(v), u \sin(v), u \rangle$ of the cone, which is rotationally symmetric.

- 6) ☒ T ☐ F If $(1, 1)$ is a critical point for the function $f(x, y)$ then $(1, 1)$ is also a critical point for the function $g(x, y) = f(x^2, y^2)$.

Solution:

If $\nabla f(1, 1) = (f_x(1, 1), f_y(1, 1)) = (0, 0)$ then also $\nabla g(1, 1) = (f_x(1, 1)2x, f_y(1, 1)2y) = (0, 0)$.

- 7)

T

F

 If $f(x, y)$ has a local maximum at $(0, 0)$ then it is possible that $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) < 0$.

Solution:

The conditions imply that $D = f_{xx}f_{yy} - f_{xy}^2 < 0$.

- 8)

T

F

 The integral $\int_0^x \int_0^y 1 \, dx dy$ computes the area of a region in the plane.

Solution:

This is not a valid double integral. The outer integral should not contain variables.

- 9)

T

F

 The function $f(x, y) = x^2 + y^4$ has a local minimum at $(0, 0)$.

Solution:

One can not use the second derivative test but the function is zero at $(0, 0)$ but positive everywhere else.

- 10)

T

F

 The integral $\int_0^1 \int_0^1 x^2 + y^2 \, dx dy$ is the volume of the solid bounded by the 5 planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and the paraboloid $z = x^2 + y^2$.

Solution:

In general $\int \int_R f(x, y) \, dy dx$ is the volume under the graph of f

- 11)

T

F

 There exists a region in the plane, which is neither a type I integral, nor a type II integral.

Solution:

Take for example an S shaped region.

- 12)

T

F

 Fubini's theorem assures that $\int_0^1 \int_0^x f(x, y) dy dx = \int_0^1 \int_0^y f(x, y) dx dy$.

Solution:

Fubini only applies to rectangular regions.

- 13)

T

F

 The function $f(x, y) = \sin(x) \cos(y)$ satisfies the partial differential equation $f_{xx} + f_{yy} = 0$.

Solution:

Just differentiate. It is a solution to the wave equation, not the Laplace equation.

- 14)

T

F

 Let $L(x, y)$ be the linearization of $f(x, y) = \sin(x(y + 1))$ at $(0, 0)$. Then, the level curves of $L(x, y)$ consist of lines.

Solution:

The function $L(x, y)$ is a linear function of the form $ax + by + c$ it has lines.

- 15)

T

F

 For any smooth function $f(x, y)$, the inequality $|\nabla f| \geq |f_x + f_y|$ is true.

Solution:

If $\nabla f = \langle a, b \rangle$, we square the claim, we get $a^2 + b^2 \geq (a + b)^2$. This is wrong for $(a, b) = (1, 1)$.

- 16)

T

F

 Any differentiable function $f(x, y)$ which satisfies the partial differential equation $\|\nabla f\|^2 = 0$ is constant.

Solution:

The condition $\|\nabla f\|^2 = 0$ implies that the gradient is zero and so that all directional derivatives are zero.

- 17)

T

F

 If $x + \sin(xy) = 1$, $dy/dx = \frac{-(1+y \cos(yx))}{(x \cos(xy))}$.

Solution:

This is implicit differentiation.

- 18)

T	F
---	---

 The directional derivative $D_v f(1, 1)$ is zero if v is a unit vector tangent to the level curve of f which goes through $(1, 1)$.

Solution:

The level curve is perpendicular to the gradient.

- 19)

T	F
---	---

 If (a, b) is a maximum of $f(x, y)$ under the constraint $g(x, y) = 0$, then the Lagrange multiplier λ there has the same sign as the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at (a, b) .

Solution:

False, by changing g to $-g$, we can change the Lagrange multiplier, but the discriminant stays the same.

- 20)

T	F
---	---

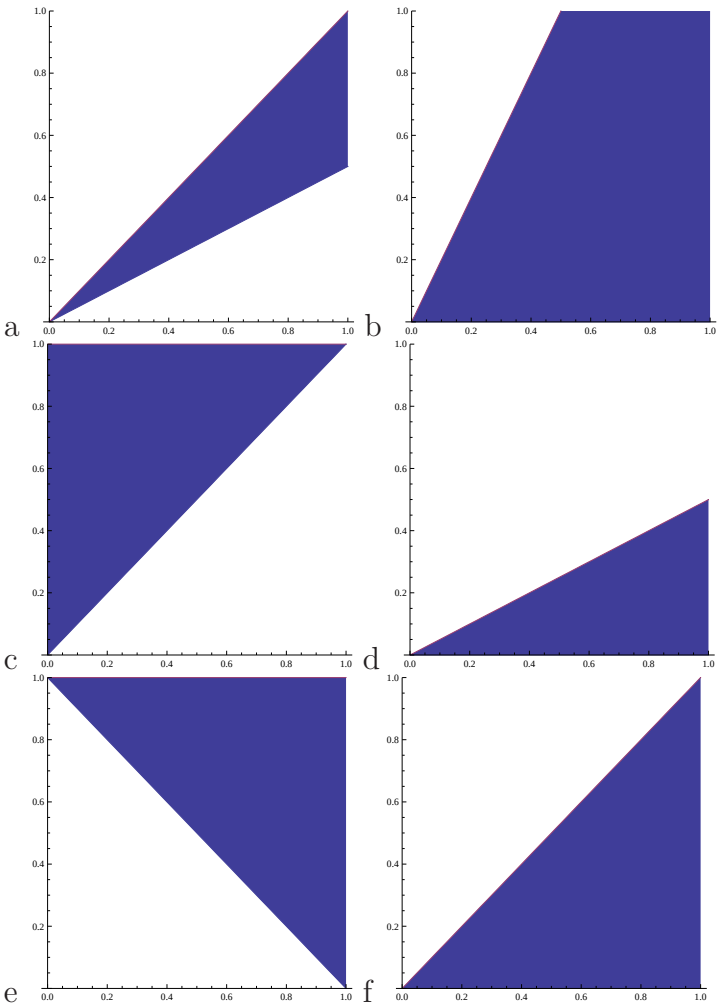
 If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(1, 2) = 0$ and $D_{\langle -1/\sqrt{2}, 1/\sqrt{2} \rangle} f(1, 2) = 0$, then $(1, 2)$ is a critical point.

Solution:

Indeed, if $\nabla f = \langle a, b \rangle$, then $\langle a, b \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = 0$ and $\langle a, b \rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = 0$ which implies $a = b = 0$.

Problem 2) (10 points)

Match the regions with the corresponding double integrals



Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
	$\int_0^1 \int_{x/2}^x f(x,y) \, dydx$
	$\int_0^1 \int_0^y f(x,y) \, dx dy$
	$\int_0^1 \int_0^{x/2} f(x,y) \, dydx$
	$\int_0^1 \int_{y/2}^1 f(x,y) \, dx dy$
	$\int_0^1 \int_0^x f(x,y) \, dydx$
	$\int_0^1 \int_{1-x}^1 f(x,y) \, dydx$

Solution:

a,c,d,b,f,e.

Problem 3) (10 points)

Let $g(x, y, z) = x^2 + 2y^2 - z - 3$.a) (5 points) Find the equation of the tangent plane to the level surface $g(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (2, 0, 1)$.b) (5 points) The surface in a) is the graph $z = f(x, y)$ of a function of two variables. Find the tangent line to the level curve $f(x, y) = 1$ at the point $(x_0, y_0) = (2, 0)$.**Solution:**a) $\nabla g(x, y, z) = \langle 2x, 4y, -1 \rangle$. $\nabla g(2, 0, 1) = \langle 4, 0, -1 \rangle$ The equation of the tangent plane is

$$4x + 0y - z = 7 .$$

where the value of 7 has been obtained by plugging in the point $(2, 0, 1)$. The final answer is $4x - z = 7$.b) $z = x^2 + 2y^2 - 3 = f(x, y)$.

$$\nabla f(x, y) = \langle 2x, 4y \rangle .$$

and

$$\nabla f(2, 0) = \langle 4, 0 \rangle .$$

The equation of the tangent line is

$$4x = 8$$

or $x = 2$.

Problem 4) (10 points)

a) (5 points) Use the technique of linear approximation to estimate $f(\pi/2 + 0.1, 2.9)$ for

$$f(x, y) = (10 \sin(x) - 5y^2 + 8)^{1/3} .$$

b) (5 points) Find the unit vector at $(\pi/2, 3)$, in the direction where the function increases fastest.

Solution:

a) $f(\pi/2, 3) = -3$ and

$$\nabla f(x, y) = \frac{1}{3}(10 \sin(x) - 5y^2 + 8)^{-2/3} \langle 10 \cos(x), -10y \rangle .$$

so that

$$\nabla f(\pi/2, 3) = \frac{1}{27} \langle 0, -30 \rangle = \langle 0, -10/9 \rangle .$$

The estimation is $-3 + 0.1 \cdot 0 - 0.1 \cdot (-10/9) = -3 + 1/9 = -26/9$.

b) It is $(0, -1)$.

Problem 5) (10 points)

The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$.

- a) (2 points) State the chain rule which applies in this situation.
- b) (4 points) Using the chain rule in a) compute the rate of change of the pressure the observer measures at time $t = 2$.
- c) (4 points) At which time t does the observer go in the direction, in which the pressure decreases most?

Solution:

a) The multivariable chain rule is

$$\frac{d}{dt}p(\vec{r}(t)) = \nabla p(\vec{r}(t)) \cdot \vec{r}'(t) .$$

b) $\nabla p(x, y, z) = \langle 2x, 2y, -3z^2 \rangle$, $\vec{r}'(t) = \langle 1, 1, -1/t^2 \rangle$. We have $\vec{r}(2) = \langle 2, 2, 1/2 \rangle$ and $\vec{r}'(2) = \langle 1, 1, -1/4 \rangle$. By the chain rule in a), we have

$$\nabla p(2, 2, 1/2) \cdot \vec{r}'(2) = \langle 4, 4, -3/4 \rangle \cdot \langle 1, 1, -1/4 \rangle = 8 + 3/16 .$$

c) The direction in which the pressure decreases most at the observers position $\vec{r}(t)$ is $-\nabla p(\vec{r}(t)) = \langle -2t, -2t, 3/t^2 \rangle$. The question is, when this vector is parallel to the velocity vector $\langle 1, 1, -1/t^2 \rangle$. If we set

$$\langle -2t, -2t, 3/t^2 \rangle = c \langle 1, 1, -1/t^2 \rangle ,$$

we get by comparing the first coordinate $c = -2t$. The third component equation reads $3/t^2 = 2t/t^2$ and gives $3 = 2t$ leading to $t = 3/2$.

Problem 6) (10 points)

The coffee chain **Astrbucks**¹ has branches at $(0, 0)$, $(0, 3)$ and $(3, 3)$ (JFK street, Church street, and Broadway) near Harvard square. A caffeine addicted [politically correct: loving] mathematician wants to rent an apartment at a location, where the sum of the squared distances $f(x, y)$ to all those shops is a local minimum. The function is

$$f(x, y) = (x-0)^2 + (y-0)^2 + (x-0)^2 + (y-3)^2 + (x-3)^2 + (y-3)^2 = 27 - 6x + 3x^2 - 12y + 3y^2.$$

- a) (5 points) Where does the mathematician have to live to locally minimize $f(x, y)$?
- b) (3 points) For every local minimum answer: Is this local minimum a **global** minimum?
- c) (2 points) Is there a global maximum to this problem? If yes, give it. If no, why not?

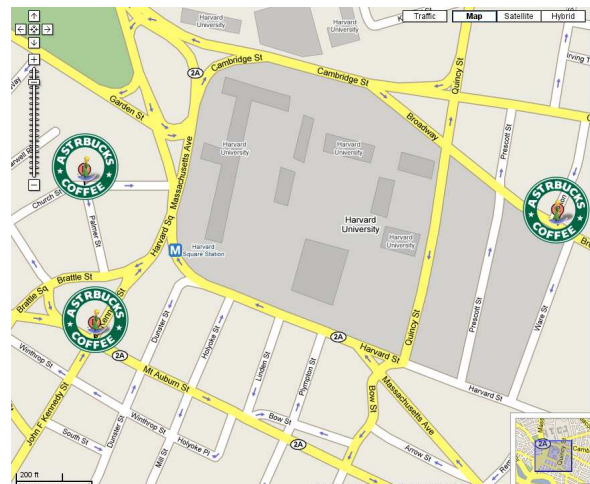
Solution:

- a) $\nabla f(x, y) = \langle 6x - 6, 6y - 12 \rangle$ is the zero vector for $(x, y) = (1, 2)$. By the second derivative test, this is a local minimum.
- b) Yes, this is a global minimum. The function can be written with a completion of squares as

$$27 - 6x + 3x^2 - 12y + 3y^2 = 12 + 3(1 - x)^2 + 3(2 - y)^2$$

which has as a graph an elliptic paraboloid with global minimum at $(1, 2)$. Many different attempts have been used here to justify the fact that we have a global minimum. Note it is not true that if a function $f(x, y)$ has one local minimum and no other critical point, then this local minimum has to be a global minimum. An example (provided by Chen-Yu Chi) is $f(x, y) = x^3 + e^{(3y)} - 3xe^y$. It has a local minimum at $(1, 0)$ but not other critical point and no global minimum nor maximum.

- c) It is possible to argue that the function increases monotonically if $x^2 + y^2$ is large enough and goes to ∞ . This prevents the existence of a global maximum.



Problem 7) (10 points)

¹This problem was sponsored by Astrbucks©.

Find all the critical points of $f(x, y) = 3xy + x^2y + xy^2$ and classify them as saddle points, local maxima or local minima.

Solution:

The gradient is

$$\nabla f(x, y) = \langle 3y + 2xy + y^2, 3x + x^2 + 2xy \rangle .$$

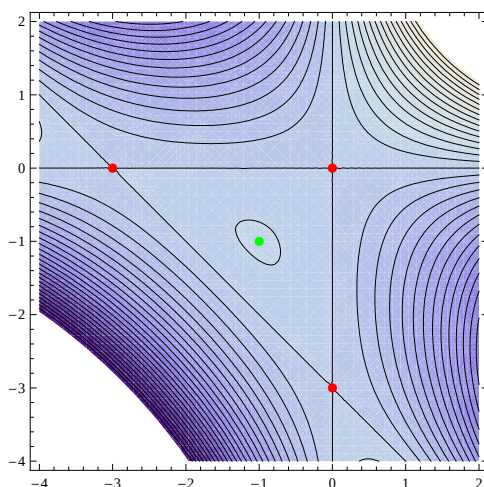
Factor out y in the first component and x in the second component.

$$\nabla f(x, y) = \langle y(3 + 2x + y), x(3 + x + 2y) \rangle = \langle 0, 0 \rangle .$$

If $y = 0$, then the first equation holds and the second equation needs either $x = 0$ or $x = -3$. If $x = 0$ then the second equation holds and the first equation needs either $y = 0$ or $y = -3$. If both x and y are not zero, then $3 + 2x + y = 0, 3 + x + 2y = 0$ which has the solution $x = y = -1$. We have therefore 4 solutions. We evaluate the discriminant $D = 4xy - (3 + 2x + 2y)^2$ and the second derivative $f_{xx} = 2y$ at each point:

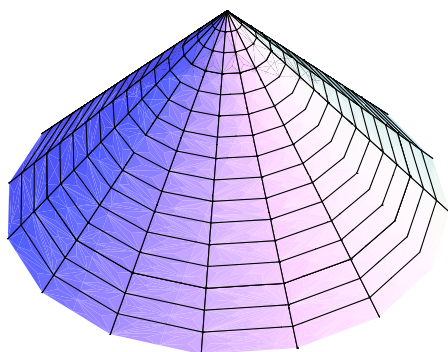
point	D	f_{xx}	nature of the critical point
$(-3, 0)$	-9	0	saddle point
$(-1, -1)$	3	-2	local max
$(0, -3)$	-9	-6	saddle point
$(0, 0)$	-9	0	saddle point

The picture shows some level curves of the function $f(x, y)$:



Problem 8) (10 points)

A solid cone of height h and with base radius r has the volume $f(h, r) = \frac{h\pi r^2}{3}$ and the surface area $g(h, r) = \pi r\sqrt{r^2 + h^2} + \pi r^2$. Among all cones with fixed surface area $g(h, r) = \pi$ use the Lagrange method to find the cone with maximal volume.



Solution:

The Lagrange equations are after dividing out π factors in all 3 equations

$$\begin{aligned} f_r = 2rh/3 &= \lambda(2r + \sqrt{r^2 + h^2} + r^2/\sqrt{r^2 + h^2}) = g_r \\ f_h = r^2/3 &= \lambda rh/\sqrt{r^2 + h^2} = g_h \\ r\sqrt{r^2 + h^2} + r^2 &= 1 \end{aligned}$$

Because $r = 0$ is incompatible with the third equation, we can divide by r in the second and third equation equation:

$$\begin{aligned} 2rh/3 &= \lambda(2r + \sqrt{r^2 + h^2} + r^2/\sqrt{r^2 + h^2}) \\ r/3 &= \lambda h/\sqrt{r^2 + h^2} \\ \sqrt{r^2 + h^2} &= \frac{1 - r^2}{r} \end{aligned}$$

Now plug the third equation into the first two to have a simpler system and also square the third equation

$$\begin{aligned} 2rh/3 &= \lambda(2r + (1 - r^2)/r + r^3/(1 - r^2)) \\ r/3 &= \lambda hr/(1 - r^2) \\ r^2 + h^2 &= \frac{(1 - r^2)^2}{r^2} \end{aligned}$$

We get rid of λ by dividing the first by the second equation

$$\begin{aligned} 2h &= 1/(hr^2) \\ r^2 + h^2 &= \frac{(1 - r^2)^2}{r^2} \end{aligned}$$

The first equation gives $h^2 = 1/(2r^2)$. Plugging this into the third gives $r = 1/2$. The solution is $\boxed{r = 1/2, h = \sqrt{2}}$.

Problem 9) (10 points)

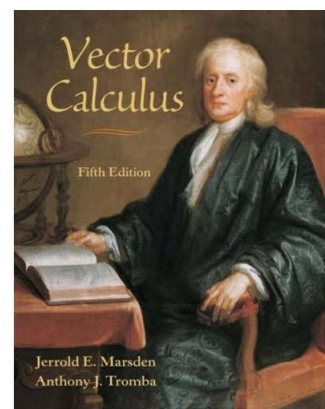
Marsden and Tromba pose in their textbook the following riddle:
Suppose $w = f(x, y)$ and $y = x^2$. By the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + 2x \frac{\partial w}{\partial y}$$

so that $0 = 2x \frac{\partial w}{\partial y}$ and so $\frac{\partial w}{\partial y} = 0$.

a) Find an explicit example of a function $f(x, y)$, where you see the argument is false.

b) What is flawed in the above application of the chain rule?



Solution:

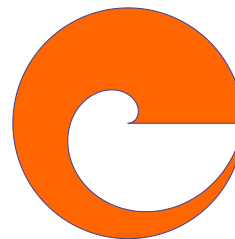
a) Take $w = f(x, y) = x^2 + y^2$ for example. Then $w_x = 2x, w_y = 2y$. b) On the left hand side of the chain rule, we have not a partial derivative any more. Lets write this more clearly. We have a function of two variables x, y and both variables x, y are functions of a third variable t . In our case $x(t) = t, y(t) = t^2$. The chain rule gives the derivative $\frac{d}{dt}f(x(t), y(t))$ as $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = f_x(t, t^2) + f_y(t, t^2)2t$. The false argument had written the left hand side of the chain rule as f_x again. Because the variable t was the variable x itself, this confusion was possible.

Problem 10) (10 points)

Evaluate the double integral

$$\int \int_R \sqrt{x^2 + y^2} \, dx dy$$

where R is the region bounded by the positive x -axes, the spiral curve $\vec{r}(t) = \langle t \cos(t), t \sin(t) \rangle, 0 \leq t \leq 2\pi$ and the circle with radius 2π .



Solution:

Use polar coordinates, where $\sqrt{x^2 + y^2} = r$ and where $dx dy$ is replaced by $r dr d\theta$.

$$\int_0^{2\pi} \int_{\theta}^{2\pi} r^2 \, dr d\theta = \int_0^{2\pi} (8\pi^3 - \theta^3)/3 \, d\theta .$$

This integral is $\boxed{4\pi^4}$.