11/7/2013 SECOND HOURLY PRACTICE II **Math 21a, Fall 2013**

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Yu-Wen Hsu
MWF 10 Yong-Suk Moon
MWF 11 Rosalie Belanger-Rioux
MWF 11 Gijs Heuts
MWF 11 Siu-Cheong Lau
MWF 12 Erick Knight
MWF 12 Kate Penner
TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) | T | F |

If $\vec{r}(t)$ is a space curve satisfying $\vec{r}\,'(0) = 0$ and $f(x, y, z)$ is a function of three variables then $\frac{d}{dt}f(\vec{r}(t)) = 0$ at $t = 0$.

Solution:
This is the chain rule

2) | T | F |

The integral $\int\int_R 1\, dxdy$ is the area of the region R in the xy-plane.

Solution:
Yes, it is the volume of the region below the function $f(x, y) = 1$.

3) | T | F |

If $f(x, y)$ is a linear function in x, y, then $D_{\vec{u}}f(x, y)$ is independent of $\vec{u}$.

Solution:
The gradient is constant $\langle a, b\rangle$ and the directional derivative $D_u f(x, y) = \langle a, b\rangle \cdot \langle \cos(t), \sin(t)\rangle$ depends on the direction.

4) | T | F |

If $f(x, y)$ is a linear function in x, y, then $D_{\vec{u}}f(x, y)$ is independent of (x, y).

Solution:
The gradient does not depend on (x, y) so that $D_u f(x, y) = \nabla f(x, y) \cdot u$ does not depend on (x, y).

5) | T | F |

If $(0, 0)$ is a saddle point of $f(x, y)$ it is possible that $(0, 0)$ is a minimum of $f(x, y)$ under the constraint $x = y$.

Solution:
The gradient is zero so that the directional derivative is zero in all direction.

6) | T | F |

The equation $f_{xy}(x, y) = 0$ is an example of a partial differential equation.

Solution:
Yes, it is an equation for a function and the equation contains partial derivatives with respect to different variables.

7) | T | **F** |

The linearization of $f(x,y) = 4 + x^3 + y^3$ at $(x_0, y_0) = (0,0)$ is $L(x,y) = 4 + 3x^2 + 3y^2$.

Solution:
The linearization is a linear function.

8) | T | **F** |

Assume $(1,1)$ is a saddle point of $f(x,y)$. Then $D_{\vec{v}} f(1,1)$ takes both positive and negative values as $\vec{v}$ varies over all directions.

Solution:
It is constant zero. Because the gradient is zero.

9) | **T** | F |

The integral $\int_{\pi/2}^{\pi} \int_0^2 r \, dr d\theta$ is equal to π.

Solution:
Yes, it is the area of a quarter of a disc of radius π.

10) | **T** | F |

If $|\nabla f(0,0)| = 1$, then there is a direction in which the slope of the graph of f at $(0,0)$ is 1.

Solution:
It is the direction of the gradient

11) | T | **F** |

The vector $\nabla f(a,b)$ is a vector in space orthogonal to surface defined by $z = f(x,y)$ at the point (a,b).

Solution:
Big misconception. This gradient vector is a vector in the plane, not in space.

12) | **T** | F |

If $f(x,y,z) = 1$ defines y as a function of x and z, then $\partial y(x,z)/\partial x = -f_x(x,y,z)/f_y(x,y,z)$.

13) T F In a constrained optimization problem it is possible that the Lagrange multiplier λ is 0.

14) T F The area $\int\int_R |\vec{r}_u \times \vec{r}_v| \, dudv$ of a surface is independent of the parametrization.

Solution:
Yes.

15) T F The function $f(x,y) = x^6 + y^6 - x^5$ has a global minimum in the plane.

16) T F The area of a graph $z = f(x,y)$ where (x,y) is in a region R is the integral $\int\int_R |f_x \times f_y| \, dxdy$.

Solution:
The equation does not make sense. f_x, f_y are both scalars, not vectors. If we want to compute the surface area, we have to parametrize the surface first as $\langle x, y, f(x,y)\rangle$.

17) T F The gradient of a function $f(x,y)$ of two variables can be written as $\langle D_{\vec{i}}f(x,y), D_{\vec{j}}f(x,y)\rangle$, where $\vec{i} = \langle 1,0\rangle$ and $\vec{j} = \langle 0,1\rangle$.

18) T F The length of the gradient of f at a critical point is positive if the discriminant $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ is strictly positive.

Solution:
The length of the gradient is zero.

19) T F If $f(0,0) = 0$ and $f(1,0) = 2$ then there is a point on the line segment between $(0,0)$ and $(1,0)$, where the gradient has length at least 2.

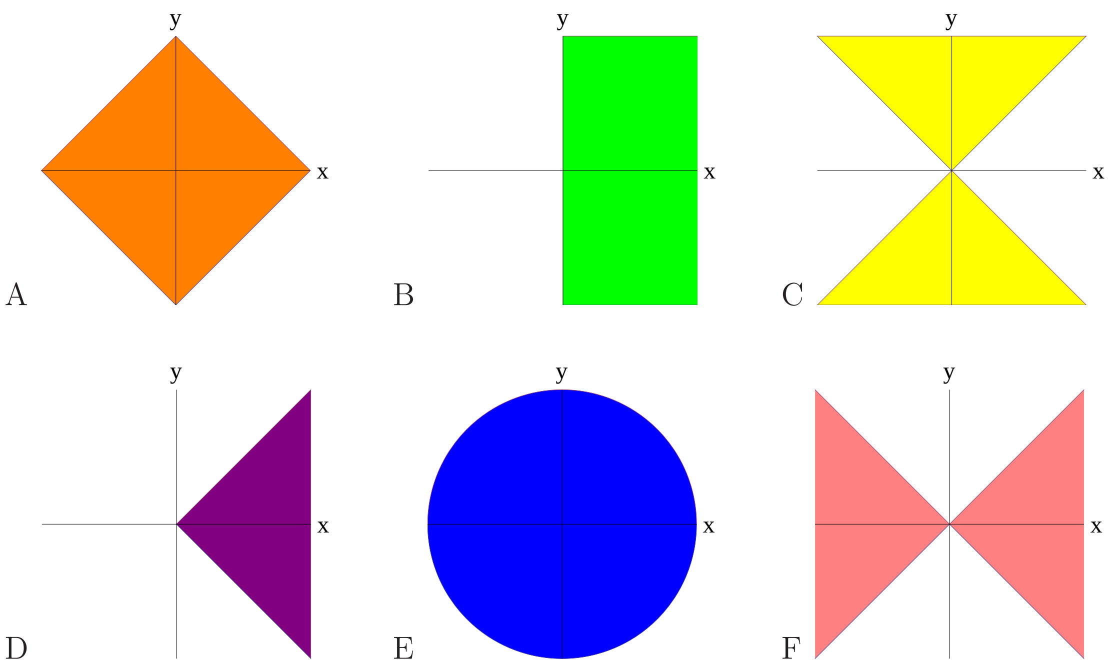
Solution:
This is the intermediate value theorem applied to the function $g(x) = f(x,0)$.

20) T F The tangent plane of the surface $-x^2 - y^2 + z^2 = 1$ at $(0,0,1)$ intersects the surface at exactly one point.

Solution:
The linearization is a linear function.

Problem 2) (10 points)

a) (6 points) Match the regions with the integrals. Each integral matches exactly one region $A - F$.



Enter A-F	Integral
	$\int_{-\pi}^{\pi} \int_{-\lvert y\rvert}^{\lvert y\rvert} f(x,y)\, dxdy$
	$\int_{-\pi}^{\pi} \int_{0}^{\pi} f(r,\theta) r\, drd\theta$
	$\int_{-\pi}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$
	$\int_{0}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$
	$\int_{-\pi}^{\pi} \int_{0}^{\pi} f(x,y)\, dxdy$
	$\int_{-\pi}^{\pi} \int_{-\pi+\lvert x\rvert}^{\pi-\lvert x\rvert} f(x,y)\, dydx$

b) (4 points) Name the partial differential equations correctly. Each equation appears once to the left.

Fill in 1-4	Order
	Burgers
	Transport
	Heat
	Wave

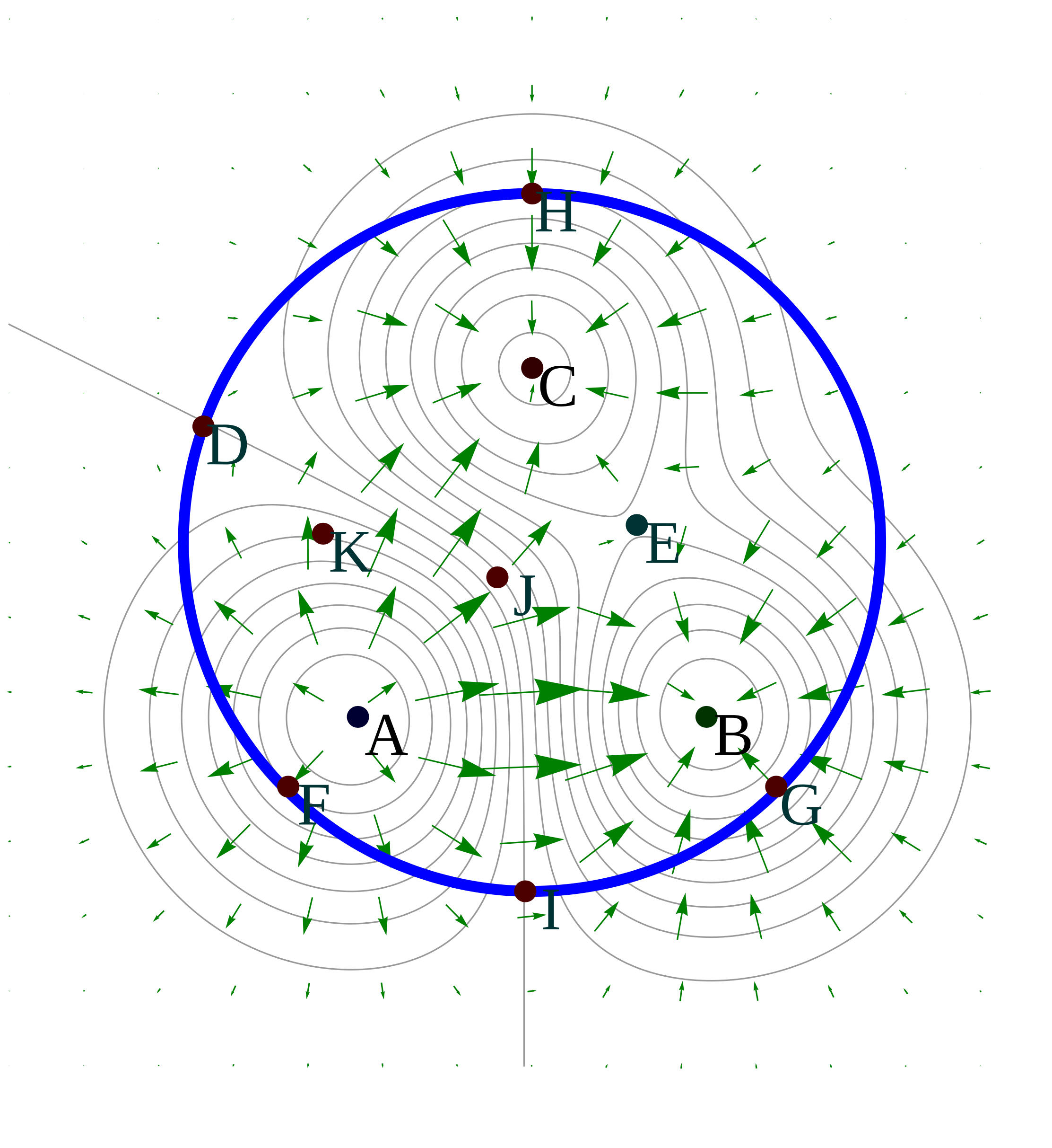
Equation Number	PDE
1	$u_x - u_y = 0$
2	$u_{xx} - u_{yy} = 0$
3	$u_x - u_{yy} = 0$
4	$u_x + uu_x - u_{xx} = 0$

Solution:
a) CEFDBA
b) 4132

Problem 3) (10 points)

(10 points) Let's label some points in the following contour map of a function $f(x, y)$ indicating the height of a region. The arrows indicate the gradient $\nabla f(x, y)$ at the point. Each of the 11 selected points appears each exactly once.

Enter A-K	description
	a local minimum of $f(x, y)$ inside the circle
	a saddle point of $f(x, y)$ inside the circle
	a point, where $f_x \neq 0$ and $f_y = 0$
	a point, where $f_x = 0$ and $f_y > 0$
	a point, where $f_x = 0$ and $f_y < 0$
	a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$.
	the lowest point on the circle
	the highest point on the circle
	the local but not global maximum inside or on the circle
	the global maximum inside or on the circle
	the steepest point inside the circle



Solution:

Enter A-K	description
A	a local minimum of $f(x,y)$ inside the circle
E	a saddle point of $f(x,y)$ inside the circle
I	a point, where $f_x \neq 0$ and $f_y = 0$
K	a point, where $f_x = 0$ and $f_y > 0$
H	a point, where $f_x = 0$ and $f_y < 0$
D	a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$.
F	the lowest point on the circle
G	the highest point on the circle
C	the local but not global maximum inside or on the circle
B	the global maximum inside or on the circle
J	the steepest point inside the circle

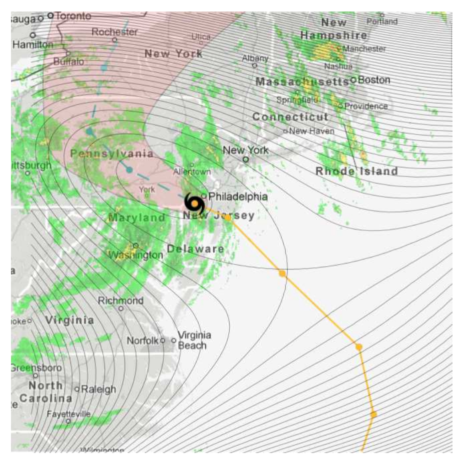
Solution:
AEIKHDFGCBJ.

Problem 4) (10 points)

On October 30, 2012, the wind speed of Hurricane Sandy was given by the function

$$f(x,y) = 60 - x^3 + 3xy + y^3 \ .$$

Classify the critical points (maxima, minima and saddle points) of this function. Compute also the values of f at these points.



Solution:
Compute the gradient $\nabla f(x,y) = \langle -3x^2+3y, 3x+3y^2\rangle$, then $f_{xx} = -6x$ and $D = f_{xx}f_{yy} - f_{xy}^2 = -36xy - 3$.

Point	D	f_{xx}	$f(x,y)$	
$(-1,1)$	27	6	*minimum*	59
$(0,0)$	-9	0	*saddle*	60

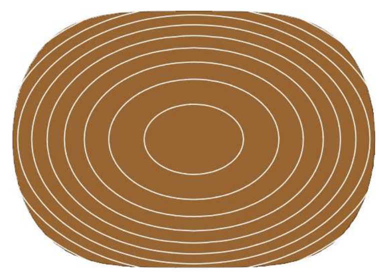
.

Problem 5) (10 points)

Use the second derivative test and the method of Lagrange multipliers to find the global maximum and minimum of the sugar concentration $f(x,y) = 10 + x^2 + 2y^2$ on a cake given by

$$g(x,y) = x^4 + 4y^2 \leq 4 .$$

Note that this means you have to look both inside the cake and on the boundary.



Solution:
(i) We compute first the critical points in the interior. To do so, we find the gradient $\nabla f(x,y) = \langle 2x, 4y\rangle$. In the interior, there is the critical point $(0,0)$ and the value of the function is 10 at this point. It is a local minimum.
(ii) Now we compute the critical points on the boundary. This is a Lagrange problem. The Lagrange equations are

$$\begin{aligned} 2x &= \lambda 4x^3 \\ 4y &= \lambda 8y . \\ x^4 + 4y^2 &= 4 . \end{aligned}$$

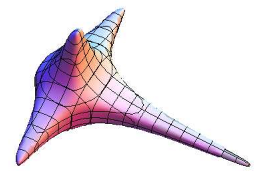
Eliminating λ from the first two equations gives $16xy = 16yx^3$. This means either $x = 0$ or $y = 0$ or $x = \pm 1$. From the third equation we can then get the value of the other variable. There are 8 critical points $(0, \pm 1)$, $(\pm 1, \pm\sqrt{3}/2)$, $(\pm\sqrt{2}, 0)$ on the boundary. The function takes the value $12, 12.5, 12$. Therefore, the global maxima are at $(\pm 1, \pm\sqrt{3}/2)$ and the global minimum is at $(0,0)$.

Problem 6) (10 points)

a) (5 points) A seed of "Tribulus terrestris" has the shape

$$x^2 + y^2 + z^2 + x^4y^4 + x^4z^4 + y^4z^4 - 9z = 21$$

Find the tangent plane at $(1, 1, 2)$.



b) (5 points) The seed intersects with the xy-plane in a curve

$$x^2 + y^2 + x^4y^4 = 21 .$$

Find the tangent line to this curve at $(1, 2)$.

Solution:
a) Compute the gradient

$$\nabla f(x, y, z) = \langle 2x + 4x^3(y^4 + z^4), 2y + 4y^3(x^4 + z^4), 2z + 4z^3(x^4 + y^4) - 9\rangle .$$

Evaluated at $(1, 1, 2)$ it is $\langle 70, 70, 59\rangle$. The equation of the plane is $70x + 70y + 59z = d$. Plug in the point $(1, 1, 2)$ to find $d = 258$ so that $\boxed{70x + 70y + 59z = 258}$.
b) Compute the gradient

$$\nabla f(x, y) = \langle 2x + 4x^3y^4, 2y + 4x^4y^3\rangle .$$

At $(1, 2)$ it is $\langle 66, 36\rangle$. The equation of the line is $66x + 36y = d$. now plug in the point to get $d = 138$. The equation is $\boxed{11x + 6y = 23}$.

Problem 7) (10 points)

Let $f(x, y)$ model the time that it takes a rat to complete a maze of length x given that the rat has already run the maze y times. We know $f_y(10, 20) = -5$ and $f_x(10, 20) = 1$ as well as $f(10, 20) = 45$. Use this to estimate $f(11, 18)$.

Picture by Ellen van Deelen,

South west News service (UK)

Solution:
Use the linearization $L(x, y)$ to the function at $(10, 20)$:

$$f(11, 18) \sim L(11, 18) = 45 + 1(11 - 10) - 5(18 - 20) = 56 \ .$$

Problem 8) (10 points)

a) (5 points) Find the double integral

$$\int \int_R x \ dydx \ ,$$

where R is the region obtained by intersecting $x \leq |y|$ with $x^2 + y^2 \leq 1$.

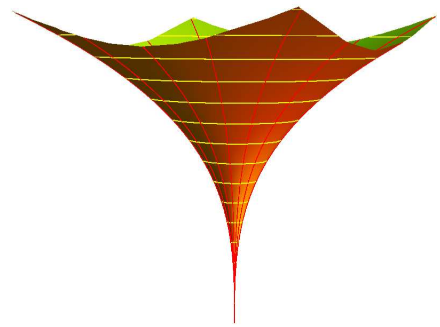
b) (5 points) The square $\sin^2(x)/x^2$ of the sync function $\sin(x)/x$ does not have a known antiderivative. Compute nevertheless the integral

$$\int_0^{\pi^2/4} \int_{\sqrt{y}}^{\pi/2} \frac{\sin^2(x)}{x^2} \ dx \ dy \ .$$

Solution:
a) Make a picture of the region. Use Polar coordinates: $\int_{\pi/4}^{7\pi/4} \int_0^1 r^2 \cos(\theta) \ drd\theta = -\sqrt{2}/3$.
b) Change the order of integration:

$$\int_0^{\pi/2} \int_0^{x^2} \frac{\sin^2(x)}{x^2} \ dydx = \frac{\pi}{4} \ .$$

Problem 9) (10 points)



Find the surface area of the surface of revolution $x^2 + y^2 = z^6$ where $0 \leq z \leq 1$. The surface is parametrized by

$$\vec{r}(t,z) = \langle z^3\cos(t), z^3\sin(t), z \rangle$$

with $0 \leq t \leq 2\pi$ and $0 \leq z \leq 1$.

Solution:
Compute

$$\vec{r}_t = \langle -z^3 \sin 9t), z^3\cos(t), 0\rangle, \vec{r}_z = \langle 3z^2\cos(t), 3z^2\sin(t), 1\rangle \ .$$

Now

$$|\vec{r}_t \times \vec{r}_z| = \sqrt{z^6\cos^2(t) + z^6\sin^2(t) + 9z^10} = z^3\sqrt{1+9z^4} \ .$$

We integrate

$$\int_0^{2\pi}\int_0^1 z^3\sqrt{1+9z^4}\, dzd\theta = (1+9z^4)^{3/2}\frac{1}{3}\frac{1}{9}2\pi =$$

$\boxed{(10^{3/2}-1)\frac{\pi}{27}.}$

Problem 10) (10 points)

It turns out that there is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!

a) (6 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{x^7}{\sqrt{x} - x^2}\, dx\, dy \ .$$

b) (4 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)^{10}\, dxdy \ .$$

Solution:

a) Change the order of integration and first switch also the order of the integral because y^2 is smaller than $\sqrt{y}$.

$$= -\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{x^7}{\sqrt{x} - x^2} \, dy \, dx = -\int_0^1 x^7 \, dx = -1/8 \; .$$

b) Write the integral in polar coordinates noticing that the region is a quarter circle.

$$\int_0^{\pi/2} \int_0^1 r^{21} \, dr \, d\theta = \frac{\pi}{44} \; .$$

The answers are $\boxed{-1/8}$ and $\boxed{\pi/44}$.